pre-knowledge：单位制.

- 国际单位制中，力学与电磁学的基本量为时间 T，长度 L，质量 M，电流 A.
- GR中常用<u>几何单位制</u>（system of geometrized units），i.e. 取 $c=G=1$.

  由于 $c=G=1$，故时间、长度、质量三者单位只有一个独立. 不妨取时间 T 为基本量，秒 s 为基本单位.

  <u>几何高斯制</u>中，再取真空介电常量 $\varepsilon_0=1$，i.e. 介电常量 ε 为基本量，ε_0 为基本单位.

- 单位制的转换.（由于 $\varepsilon_0=G=c=1$，信息被压缩；转换过程即信息的还原过程）

  记某一物理量为 $\tilde{p}$，在国际制中数值为 p，量纲为 $[\tilde{p}]$. 几何制中数值为 p'，量纲为 $[\tilde{p}]'$

  几何制中，由于 $c=G=\varepsilon_0=1$，有 $p'=p'c^{\lambda}G^{\mu}\varepsilon_0^{\nu}$，于是 $[\tilde{p}]'$ <u>是有自由度的</u>.

  在国际制中，没有信息压缩，$[\tilde{p}]$ 是确定的. 通过选取 λ,μ,ν s.t. $[\tilde{p}]'=[\tilde{p}]$，则有 $p=p'c^{\lambda}G^{\mu}\varepsilon_0^{\nu}$

- 例子.

  (1) $r_s'=2M$.

  （国际制中）$[r_s'][c]^{\lambda}[G]^{\mu}=[M]^{1-\mu}[L]^{3\mu+\lambda}[T]^{-\lambda-2\mu}=[r_s]=[L] \Rightarrow \begin{cases}\mu=1\\ \lambda=-2\end{cases}$，$r_s=\dfrac{2GM}{c^2}$.

  (2) 普朗克制中，$\hbar=1$. 该制下有 $\underbrace{E'}_{\text{光子能量}}=2\pi\underbrace{\nu'}_{\text{频率}}$.

  （国际制中）$[\frac{E}{\nu}]=[T]^{-1}[M][L]^2=[\hbar]$，故 $E=2\pi\hbar\nu$.

  (3) $E'^2=m'^2+p'^2$

  $[E']=[M]$，$[m']=[M]$，$[p']=[M]$
  $[E]=[M][c]^2$，$[m]=[M]$，$[P]=[M][c]$ $\Rightarrow \begin{cases} m=m' \\ P=P'c \\ E=E'c^2 \end{cases} \Rightarrow E^2=m'^2c^4+p'^2c^4=m^2c^4+p^2c^2$.

Dirac GR

1. Coordinate transformation and tensor

$$A^{\mu'} = x^{\mu'}{}_{,\nu} A^{\nu} \equiv \frac{\partial x^{\mu'}}{\partial x^{\nu}} A^{\nu}; \quad B_{\mu'} = x^{\lambda}{}_{,\mu'} B_{\lambda} \equiv \frac{\partial x^{\lambda}}{\partial x^{\mu'}} B_{\lambda}.$$

(controvariant) (covariant)

Higher-rank tensor, eg. $T^{\alpha'\beta'}{}_{\gamma'} = x^{\alpha'}{}_{,\lambda} x^{\beta'}{}_{,\mu} x^{\nu}{}_{,\gamma'} T^{\lambda\mu}{}_{\nu}$

- for nontensors, $g^{\mu\nu}$ or $g_{\mu\nu}$ still can be used to raise / lower suffixes; contraction still works.

- the Quotient Thm: Suppose $P_{\lambda\mu\nu}$ is s.t. $A^{\lambda}P_{\lambda\mu\nu}$ is a tensor $\forall A^{\lambda}$, then $P_{\lambda\mu\nu}$ is a tensor.

(Extension is straightforward)

Pf: write $A^{\lambda}P_{\lambda\mu\nu} \equiv Q_{\mu\nu}$.

$Q_{\mu\nu}$ is a tensor, $Q_{\mu'\nu'} = x^{\rho}{}_{,\mu'} x^{\sigma}{}_{,\nu'} Q_{\rho\sigma}$

$$\begin{array}{c} \parallel \\ A^{\lambda'} P_{\lambda'\mu'\nu'} \end{array} = x^{\rho}{}_{,\mu'} x^{\sigma}{}_{,\nu'} \underset{A^{\gamma} = x^{\gamma}{}_{,\lambda'} A^{\lambda'}}{A^{\gamma}} P_{\gamma\rho\sigma} \Rightarrow P_{\lambda'\mu'\nu'} = x^{\gamma}{}_{,\lambda'} x^{\rho}{}_{,\mu'} x^{\sigma}{}_{,\nu'} P_{\gamma\rho\sigma} \quad \square.$$

2. Parallel displacement

- In a curved space, meaning of "parallel" is ambigouous.

But if we take a point P' close to P, there is a paralled vector at P' with an uncertainty of the 2nd order.

(counting $\overline{PP'}$ as the 1st order)

With this process, we can end up with a vector at Q paralled to the original vector at P wrt a chosen path.

- Calculating paraller displacement by immerse dim-4 physical space in a flat space of dim-N ($N > 4$).

In dim-N flat space, introduce rectilinear coordinate (直角坐标) z^n ($n = 1, \dots, N$).

Then $ds^2 = h_{nm} z^n z^m$, $h_{nm} = \text{const}$. def $dz_n = h_{nm} dz^m$.

since dim-4 physical space is a "surface" in this dim-N flat space,

each point x^{μ} on the surface determines a definite point $y^n(x)$ in dim-N space.

for two neighboring points in the surface differing by δx^{μ}, $\delta y^n = y^n{}_{,\mu} \delta x^{\mu}$; $y^n{}_{,\mu} \equiv \dfrac{\partial y^n}{\partial x^{\mu}}$.

the squre distance: $\delta s^2 = h_{nm} y^n y^m = h_{nm} y^n{}_{,\mu} y^m{}_{,\nu} \delta x^{\mu} \delta x^{\nu} = y^n{}_{,\mu} y_{n,\nu} \delta x^{\mu} \delta x^{\nu}$

in the surface, $\delta s^2 = g_{\mu\nu} \delta x^{\mu} \delta x^{\nu}$.

$$\Rightarrow g_{\mu\nu} = h_{nm} y^n{}_{,\mu} y^m{}_{,\nu} = y^n{}_{,\mu} y_{n,\nu}. \quad * \ n, m = 1, \dots, N; \ \mu, \nu = 0, 1, 2, 3.$$

for A^{μ} in physical space, it corresponds with a vector A^n in dim-N space: $A^n = y^n{}_{,\mu} A^{\mu}$.

- Shift A^n parallelly to a point $(x + dx)$ in the surface, then it no longer lie in the surface.

(A^n 仍是平面上某点的向量，但不再在平面中，i.e. 不再切于该平面)

Split $A^n = A^n{}_{tan} + A^n{}_{nor}$, let K^{μ} denotes the components of $A^n{}_{tan}$ referred to the x coordinate system in the surface

$$\Rightarrow A^n{}_{tan} = K^{\mu} y^n{}_{,\mu}(x + dx); \quad h_{nm} A^n{}_{nor} A^m{}_{tan} = A^n{}_{nor} A_{n,tan} = A^n{}_{nor}\, y_{n,\mu}(x + dx)\, K^{\mu} = 0, \ \forall K^{\mu}.$$

After displacement, $A^n y_{n,\nu}(x+dx) = A^n_{\tan} y_{n,\nu}(x+dx) = K^\mu y^n{}_{,\mu}(x+dx)\, y_{n,\nu}(x+dx) = K^\mu g_{\mu\nu}(x+dx) \equiv K_\nu(x+dx)$

To the 1st order, $K_\nu(x+dx) = A^n[y_{n,\nu}(x) + y_{n,\nu,\sigma} dx^\sigma]$

$$= A^\mu y^n{}_{,\mu}[y_{n,\nu} + y_{n,\nu,\sigma} dx^\sigma]$$

$$= A_\nu + A^\mu y^n{}_{,\mu} y_{n,\nu,\sigma} dx^\sigma$$

then the change dA_ν under parallel displacement: $dA_\nu \equiv K_\nu - A_\nu = A^\mu y^n{}_{,\mu} y_{n,\nu,\sigma} dx^\sigma$.

3. Christoffel symbols.

- $g_{\mu\nu} = y^n{}_{,\mu} y_{n,\nu} \xrightarrow{\partial/\partial x^\sigma} g_{\mu\nu,\sigma} = y^n{}_{,\mu\sigma} y_{n,\nu} + y^n{}_{,\mu} y_{n,\nu\sigma}$

$$= \underline{y_{n,\mu\sigma} y^n{}_{,\nu}} + y_{n,\mu} y^n{}_{,\nu\sigma}$$

$\mu \leftrightarrow \sigma$: $g_{\sigma\nu,\mu} = \underline{y_{n,\sigma\mu} y^n{}_{,\nu}} + \underline{y_{n,\sigma} y^n{}_{,\nu\mu}}$

$\nu \leftrightarrow \sigma$: $g_{\mu\sigma,\nu} = \underline{y_{n,\mu\nu} y^n{}_{,\sigma}} + y_{n,\mu} y^n{}_{,\sigma\nu}$

$$\Rightarrow \Gamma_{\mu\nu\sigma} \equiv \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) = y^n{}_{,\nu\sigma} y_{n,\mu}.$$

$\longrightarrow$ Christoffel symbol of the first kind.

- $\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} = g_{\mu\nu,\sigma}$. All ref. to the dim-N space disappear!
- $\Gamma_{\mu\nu\sigma} = \Gamma_{\mu\sigma\nu}$, $\Gamma^\mu{}_{\nu\sigma} = \Gamma^\mu{}_{\sigma\nu}$.
- $dA_\nu = A^\mu \Gamma_{\mu\nu\sigma} dx^\sigma = \Gamma^\mu{}_{\nu\sigma} A_\mu dx^\sigma$
- the length of a vector is unchanged by parallel displacement.

Pf: $d(g^{\mu\nu}A_\mu A_\nu) = g^{\mu\nu}A_\mu dA_\nu + g^{\mu\nu}A_\nu dA_\mu + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma} dx^\sigma$

$$= A^\nu dA_\nu + A^\mu dA_\mu + A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma} dx^\sigma$$

$$= A^\nu A^\mu \Gamma_{\mu\nu\sigma} dx^\sigma + A^\mu A^\nu \Gamma_{\nu\mu\sigma} dx^\sigma + A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma} dx^\sigma$$

$$= A^\nu A^\mu g_{\mu\nu,\sigma} dx^\sigma + A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma} dx^\sigma$$

Consider $g^{\alpha\mu}{}_{,\sigma} g_{\mu\nu} + g^{\alpha\mu} g_{\mu\nu,\sigma} = (g^{\alpha\mu} g_{\mu\nu})_{,\sigma} = \delta^\alpha{}_{\nu,\sigma} = 0$.

$\times g^{\beta\nu}$: $g^{\beta\nu} g^{\alpha\mu}{}_{,\sigma} g_{\mu\nu} + g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma} = \delta^\beta_\mu g^{\alpha\mu}{}_{,\sigma} + g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma} = 0$

- $g^{\alpha\beta}{}_{,\sigma} = -g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma}$, a useful formula giving $g^{\alpha\beta}$ in terms of $g_{\mu\nu}$.

$$\Rightarrow A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma} = -A^\mu A^\nu g_{\mu\nu,\sigma} \Rightarrow d(g^{\mu\nu} A_\mu A_\nu) = 0. \quad \square$$

- $\Gamma^\mu{}_{\nu\sigma} = g^{\mu\lambda} \Gamma_{\lambda\mu\sigma}$ ~ the Christoffel symbol of the 2nd kind
- $dB^\nu = -\Gamma^\nu{}_{\mu\sigma} B^\mu dx^\sigma$, parallel displacement formula refering to contravariant components

Pf: $d(A_\nu B^\nu) = 0 \rightarrow \underline{A_\nu} dB^\nu = -B^\nu dA_\nu = -B^\nu \Gamma^\mu{}_{\nu\sigma} A_\mu dx^\sigma = -B^\mu \Gamma^\nu{}_{\mu\sigma} \underline{A_\nu} dx^\sigma$. $\square$

4. Geodesics

Suppose a point z^μ moves along a track, parameterized by τ, i.e. $z^\mu = z^\mu(\tau)$

def $u^\mu \equiv \frac{dz^\mu}{d\tau}$, called the tangent vector.

- Given an initial point z^μ and initial value of u^μ, shift z^μ to $z^\mu + u^\mu d\tau$,

  then shift u^μ to this new point by parallel displacement. Repeat this process.

  Then the track is determined, so does τ along it. This track is a geodesic.

- if initial u^μ is a null vector $\to$ null geodesic. (the path of light is a null geodesic)

  timelike $\to$ timelike geodesic

  spacelike $\to$ spacelike geodesic

- From $dB^\nu = -\Gamma^\nu_{\mu\sigma} B^\mu dx^\sigma$, substitute $B^\nu = u^\nu$, $dx^\sigma = dz^\sigma$

$$\Rightarrow \frac{du^\nu}{d\tau} + \Gamma^\nu_{\mu\sigma} u^\nu \frac{dz^\sigma}{d\tau} = 0 \iff \frac{d^2 z^\nu}{d\tau} + \Gamma^\nu_{\mu\sigma} \frac{dz^\nu}{d\tau}\frac{dz^\sigma}{d\tau} = 0.$$

假设不受引力以外的任何力的质点，其世界线为类时测地线。该方程可作为该质点的运动方程。

Choose the para. τ as the proper time s, then $u^\mu \to v^\mu = \frac{dz^\mu}{ds}$ becomes the velocity.

$$\Rightarrow \underline{\underline{\frac{dv^\mu}{ds} + \Gamma^\mu_{\nu\sigma} v^\nu v^\sigma = 0}} \iff \frac{d^2 z^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}\frac{dz^\nu}{ds}\frac{dz^\sigma}{ds} = 0$$

! $g^{\mu\nu} dv^\mu dv^\nu = 1$. (for $g_{\mu\nu} dx^\mu dx^\nu = ds^2$. devide the eqn by ds^2).

- stationary properties: fixed endpoints P and Q of the track, then $\delta \int ds = 0$.

(Another derive of geodesic condiction)

Suppose each point of the track with coordinates z^μ, is shifted to $z^\mu + \delta z^\mu$;

dx^μ denotes an element along the track.

$$\Rightarrow ds^2 = g_{\mu\nu} dx^\mu dx^\nu \to 2ds\,\delta(ds) = dx^\mu dx^\nu g_{\mu\nu,\lambda}\delta x^\lambda + 2 g_{\mu\lambda} dx^\mu \underbrace{\delta dx^\lambda}_{= d\,\delta x^\lambda}$$

$dx^\mu = v^\mu ds$

$$\Rightarrow \delta(ds) = \left(\tfrac{1}{2} g_{\mu\nu,\lambda} v^\mu v^\nu \delta x^\lambda + g_{\mu\lambda} v^\mu \frac{d\,\delta x^\lambda}{ds}\right) ds$$

$$\Rightarrow \delta \int ds = \int \delta(ds) = \int \left[\tfrac{1}{2} g_{\mu\nu,\lambda} v^\mu v^\nu \delta x^\lambda + g_{\mu\lambda} v^\mu \frac{d\delta x^\lambda}{ds}\right] ds$$

$$\simeq \int \left[\tfrac{1}{2} g_{\mu\nu,\lambda} v^\mu v^\nu - \frac{d}{ds}(g_{\mu\lambda} v^\mu)\right] \delta x^\lambda ds = 0$$

$$\Rightarrow \frac{d}{ds}(g_{\mu\lambda} v^\mu) - \tfrac{1}{2} g_{\mu\nu,\lambda} v^\mu v^\nu = 0 \quad ①$$

for $\frac{d}{ds}(g_{\mu\lambda} v^\mu) = g_{\mu\lambda}\frac{dv^\mu}{ds} + g_{\mu\lambda,\nu} v^\nu v^\mu = g_{\mu\nu}\frac{dv^\mu}{ds} + \tfrac{1}{2}(g_{\mu\lambda,\nu} + g_{\lambda\mu,\nu}) v^\mu v^\nu$

$$\Rightarrow ① \iff g_{\mu\lambda}\frac{dv^\mu}{ds} + \Gamma_{\lambda\mu\nu} v^\mu v^\nu = 0$$

$\times g^{\lambda\sigma}$

$$\underline{\underline{\frac{dv^\sigma}{ds} + \Gamma^\sigma_{\mu\nu} v^\mu v^\nu = 0.}}$$

5. Covariant Differentiation

- $A_{\mu,\nu}$ is not a tensor, for $A_{\mu',\nu'} = (A_\rho x^\rho{}_{,\mu'})_{,\nu'} = A_{\rho,\sigma} x^\rho{}_{,\mu'} x^\sigma{}_{,\nu'} + A_\rho x^\rho{}_{,\mu'\nu'}$

- Modify the process of differentiation:

  take A_μ at point x, shift it to $x+dx$ by parallel displacement, $A_\mu(x) \to A_\mu(x) + \Gamma^\alpha_{\mu\nu} A_\alpha dx^\nu$

  substrat it from A_μ at $x+dx$, the difference will be a vector:

  $$A_\mu(x+dx) - [A_\mu(x) + \Gamma^\alpha_{\mu\nu} A_\alpha dx^\nu] \underset{\uparrow \text{ 1st order}}{=} (A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha) dx^\nu, \ \forall\, dx^\nu \xrightarrow[\text{thm}]{\text{Quotient}} A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha \text{ is a tensor.}$$

- Def covariant derivative of A_μ, $A_{\mu:\nu} \equiv A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha$

  - $(A_\mu B_\nu)_{:\sigma} = A_{\mu:\sigma} B_\nu + A_\mu B_{\nu:\sigma}$

    $= (A_{\mu,\sigma} - \Gamma^\alpha_{\mu\sigma} A_\alpha) B_\nu + A_\mu (B_{\nu,\sigma} - \Gamma^\alpha_{\nu\sigma} B_\alpha)$

    $= (A_\mu B_\nu)_{,\sigma} - \Gamma^\alpha_{\mu\sigma} A_\alpha B_\nu - \Gamma^\alpha_{\nu\sigma} A_\mu B_\alpha$

  $\Rightarrow T_{\mu\nu:\sigma} = T_{\mu\nu,\sigma} - \Gamma^\alpha_{\mu\sigma} T_{\alpha\nu} - \Gamma^\alpha_{\nu\sigma} T_{\mu\alpha}$, for $T_{\mu\nu}$ is expensible as a sum of terms like $A_\mu B_\nu$

  - $Y_{\mu\nu\ldots:\sigma} = Y_{\mu\nu\ldots,\sigma}$ $- \Gamma$ term for each suffix. Specially, $Y_{:\sigma} = Y_{,\sigma}$.

  - $g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - \Gamma^\alpha_{\mu\sigma} g_{\alpha\nu} - \Gamma^\alpha_{\nu\sigma} g_{\mu\alpha} = g_{\mu\nu,\sigma} - \Gamma_{\nu\mu\sigma} - \Gamma_{\mu\nu\sigma} = 0$.

    i.e. $g_{\mu\nu}$ count as const under covariant differentiation $\longrightarrow$ 可以用 $g_{\mu\nu}$, $g^{\mu\nu}$ 在协变微商前升/降指标)

  - Covariant derivative of A^μ, $A^\mu{}_{:\sigma} = A^\mu{}_{,\sigma} + \Gamma^\mu_{\alpha\sigma} A^\alpha$

    Pf: $(A^\mu B_\mu)_{:\sigma} = A^\mu{}_{:\sigma} B_\mu + A^\mu B_{\mu:\sigma} = A^\mu{}_{:\sigma} B_\mu + A^\mu (B_{\mu,\sigma} - \Gamma^\alpha_{\mu\nu} B_\alpha)$

    $(A^\mu B_\mu)_{,\sigma} = A^\mu{}_{,\sigma} B_\mu + A^\mu B_{\mu,\sigma} = (A^\mu B_\mu)_{:\sigma}$

    $\Rightarrow A^\mu{}_{,\sigma} B_\mu = A^\mu{}_{:\sigma} B_\mu - A^\mu \Gamma^\alpha_{\mu\nu} B_\alpha = A^\mu{}_{:\sigma} B_\mu - A^\alpha \Gamma^\mu_{\alpha\nu} B_\mu \Rightarrow A^\mu{}_{:\sigma} B_\mu = (A^\mu{}_{,\sigma} + \Gamma^\mu_{\alpha\nu} A^\alpha) B_\mu$. $\forall B_\mu$. $\square$

  - Laws of physics is invariant in all system of coordinates.

    The field eqns, if involve derivative, the derivative must be a covariant one.

    eg. D'Alembert eqn: $\square V = 0 \Leftrightarrow g^{\mu\nu} V_{:\mu:\nu} = 0 \longrightarrow g^{\mu\nu} (V_{,\mu\nu} - \Gamma^\alpha_{\mu\nu} V_{,\alpha}) = 0$

  - $A_{\mu:\nu} - A_{\nu:\mu} = A_{\mu,\nu} - \Gamma^\rho_{\mu\nu} A_\rho - (A_{\nu,\mu} - \Gamma^\rho_{\nu\mu} A_\rho) = A_{\mu,\nu} - A_{\nu,\mu}$.

    ~ covariant curl = ordinary curl. it holds only for a covariant vector.

6. The curvature tensor

- Order doesn't matter when performing two diff. in succession for ordinary diff., but for covariant diff it doesn't hold!
- for a scalar field S. $S_{:\mu:\nu} = S_{:\mu,\nu} - \Gamma^\alpha_{\mu\nu} S_{:\alpha} = S_{,\mu\nu} - \Gamma^\alpha_{\mu\nu} S_{,\alpha}$, in this case μ,ν are symmetric, diff order doesn't matter.

  for a vector A_μ. $A_{\nu:\rho:\sigma} = A_{\nu:\rho,\sigma} - \Gamma^\alpha_{\nu\sigma} A_{\alpha:\rho} - \Gamma^\alpha_{\rho\sigma} A_{\nu:\alpha}$

$$= (A_{\nu,\rho} - \Gamma^\alpha_{\nu\rho} A_\alpha)_{,\sigma} - \Gamma^\alpha_{\nu\sigma}(A_{\alpha,\rho} - \Gamma^\beta_{\alpha\rho} A_\beta) - \Gamma^\alpha_{\rho\sigma}(A_{\nu,\alpha} - \Gamma^\beta_{\nu\alpha} A_\beta)$$

$$= A_{\nu,\rho\sigma} - \Gamma^\alpha_{\nu\rho} A_{\alpha,\sigma} - \Gamma^\alpha_{\nu\sigma} A_{\alpha,\rho} - \Gamma^\alpha_{\rho\sigma} A_{\nu,\alpha} - A_\beta(\Gamma^\beta_{\nu\rho,\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\beta_{\alpha\rho} - \Gamma^\alpha_{\rho\sigma}\Gamma^\beta_{\nu\alpha})$$

$$A_{\nu:\sigma:\rho} = A_{\nu,\sigma\rho} - \Gamma^\alpha_{\nu\sigma} A_{\alpha,\rho} - \Gamma^\alpha_{\nu\rho} A_{\alpha,\sigma} - \Gamma^\alpha_{\sigma\rho} A_{\nu,\alpha} - A_\beta(\Gamma^\beta_{\nu\sigma,\rho} - \Gamma^\alpha_{\nu\rho}\Gamma^\beta_{\alpha\sigma} - \Gamma^\alpha_{\sigma\rho}\Gamma^\beta_{\nu\alpha})$$

$$\Rightarrow A_{\nu:\rho:\sigma} - A_{\nu:\sigma:\rho} = A_\beta \underbrace{(\Gamma^\beta_{\nu\sigma,\rho} - \Gamma^\beta_{\nu\rho,\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\beta_{\alpha\rho} - \Gamma^\alpha_{\nu\rho}\Gamma^\beta_{\alpha\sigma})}_{R^\beta_{\nu\rho\sigma}} \equiv A_\beta R^\beta_{\nu\rho\sigma}$$

Def Riemann-Choristoffel tensor / the curvature tensor : $R^\beta_{\nu\rho\sigma}$

- Obviously, $R^\beta_{\nu\rho\sigma} = -R^\beta_{\nu\sigma\rho}$

  $R^\beta_{\nu\rho\sigma} + R^\beta_{\rho\sigma\nu} + R^\beta_{\sigma\nu\rho} = 0$.

- lower the suffix β : $R_{\mu\nu\rho\sigma} = g_{\mu\beta} R^\beta_{\nu\rho\sigma} = g_{\mu\beta}\Gamma^\beta_{\nu\sigma,\rho} + \Gamma^\alpha_{\nu\sigma}\Gamma_{\mu\alpha\rho} - \langle\rho\sigma\rangle$

  * $\langle\rho\sigma\rangle$ denote the preceding terms with ρ,σ interchanged.

$$\Rightarrow R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma,\rho} - g_{\mu\beta,\rho}\Gamma^\beta_{\nu\sigma} + \Gamma^\beta_{\nu\sigma}\Gamma_{\mu\beta\rho} - \langle\rho\sigma\rangle$$

$g_{\mu\beta,\rho} = \Gamma_{\mu\beta\rho} + \Gamma_{\beta\mu\rho}$

$$= \Gamma_{\mu\nu\sigma,\rho} - \Gamma^\beta_{\nu\sigma}\Gamma_{\beta\nu\rho} - \langle\rho\sigma\rangle$$

$\begin{cases} \Gamma_{\mu\nu\sigma,\rho} = \frac{1}{2}(g_{\mu\nu,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho}) \\ \Gamma_{\mu\nu\rho,\sigma} = \frac{1}{2}(g_{\mu\nu,\rho\sigma} + g_{\mu\rho,\nu\sigma} - g_{\nu\rho,\mu\sigma}) \end{cases}$

$$= \frac{1}{2}(g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\nu\rho,\mu\sigma}) + \Gamma^\beta_{\nu\rho}\Gamma_{\beta\mu\sigma} - \Gamma^\beta_{\nu\sigma}\Gamma_{\beta\mu\rho}$$

- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$

  $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = R_{\sigma\rho\nu\mu}$

* With all symmetries, of the 256 components of $R_{\mu\nu\rho\sigma}$, only 20 are indep.

Pf: $\begin{cases} R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} & ① \\ R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} & ② \\ R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} & ③ \\ \text{cyclic symmetry} & ④ \end{cases}$

- $R_{iiii} = 0$ (Σ \)

On the basis of # indep. suffixes, deviding suffixes into 3 types.
If # indep. suffix = 1, 2, 3, then ④ reduces to ①. dims $\equiv N$.

Type 1: 2 suffixes vary. $N_I = \frac{N(N-1)}{2}$

Type 2: 3 suffixes vary. $N_{II} = \frac{N(N-1)}{2}(N-2)$

Type 3: 4 suffixes vary. $N_{III} = \frac{N(N-1)}{2}\frac{(N-2)(N-3)}{2} \cdot \frac{2}{3}$ ④

Totally, $N_t = N_I + N_{II} + N_{III} = \frac{N^2(N^2-1)}{12}$. $N_t|_{N=4} = 20$

Q.P22

- $R_{\mu\nu\rho\sigma} = 0 \Longleftrightarrow$ Space is flat. Choose a rectilinear coordinate system, then $g_{\mu\nu} = \text{const}$.

- The Bianci relations : $R^\alpha{}_{\mu\rho\sigma:\tau}+R^\alpha{}_{\mu\sigma\tau:\rho}+R^\alpha{}_{\mu\tau\rho:\sigma}=0$.

pf: write $T_{\mu\tau}$ ≡ Sum of terms like $A_\mu B_\tau$

$$(A_\mu B_\tau)_{:\rho:\sigma}=A_{\mu:\rho:\sigma}B_\tau+A_{\mu:\rho}B_{\tau:\sigma}+A_{\mu:\sigma}B_{\tau:\rho}+A_\mu B_{\tau:\rho:\sigma}$$

$A_{\nu:\rho:\sigma}-A_{\nu:\sigma:\rho}=A_\beta R^\beta{}_{\nu\sigma\rho}$ ↳ $(A_\mu B_\tau)_{:\rho:\sigma}-(A_\mu B_\tau)_{:\sigma:\rho}=A_\alpha B_\tau R^\alpha{}_{\mu\rho\sigma}+A_\mu B_\alpha R^\alpha{}_{\tau\rho\sigma}$

$$\xrightarrow{A_\mu B_\tau\to T_{\mu\tau}\to A_{\mu:\tau}} A_{\mu:\tau:\rho:\sigma}-A_{\mu:\tau:\sigma:\rho}=A_{\alpha:\tau}R^\alpha{}_{\mu\rho\sigma}+A_{\mu:\alpha}R^\alpha{}_{\tau\rho\sigma}$$

Add up cyclic permutations of τ,ρ,σ : $LHS=A_{\mu:\rho:\sigma:\tau}-A_{\mu:\sigma:\rho:\tau}$ + cyc perm

$=(A_\alpha R^\alpha{}_{\mu\rho\sigma})_{:\tau}$ + cyc perm

$=A_{\alpha:\tau}R^\alpha{}_{\mu\rho\sigma}+A_\alpha R^\alpha{}_{\mu\rho\sigma:\tau}$ + cyc perm

$R^\alpha{}_{\tau\rho\sigma}$ + cyc perm $=0$ → $RHS=A_{\alpha:\tau}R^\alpha{}_{\mu\rho\sigma}$ + cyc term

} $A_\alpha R^\alpha{}_{\mu\rho\sigma:\tau}$ + cyc perm $=0$. □.

▷ The Ricci Tensor : $R_{\nu\rho}\equiv R^\mu{}_{\nu\rho\mu}$

- $R_{\nu\rho}=R_{\rho\nu}$. pf: $R_{\mu\nu\rho\sigma}=R_{\sigma\rho\nu\mu}\xrightarrow{\times g^{\mu\sigma}}$ □.

▷ Def the scalar curvature/ total curvature: $R\equiv g^{\nu\rho}R_{\nu\rho}=R^\nu{}_\nu$

- $2R^\alpha{}_{\sigma:\alpha}-R_{:\sigma}=0$. ~ the Bianci relation of the Ricci tensor.

pf: With $R^\alpha{}_{\mu\rho\sigma:\tau}+R^\alpha{}_{\mu\sigma\tau:\rho}+R^\alpha{}_{\mu\tau\rho:\sigma}=0$, put $\tau=\alpha$, multiply by $g^{\mu\rho}$.

$g_{\mu\rho:\sigma}=0$

$$\Rightarrow \underbrace{(g^{\mu\rho}R^\alpha{}_{\mu\rho\sigma})}_{\substack{=g^{\mu\rho}g^{\alpha\beta}R_{\beta\mu\rho\sigma}\\ =g^{\alpha\beta}R_{\rho\sigma}=R^\alpha{}_\sigma}}{}_{:\alpha}+\underbrace{(g^{\mu\rho}R^\alpha{}_{\mu\sigma\alpha})}_{\substack{=g^{\mu\rho}R_{\mu\sigma}\\ =R^\rho{}_\sigma}}{}_{:\rho}+\underbrace{(g^{\mu\rho}R^\alpha{}_{\mu\alpha\rho})}_{\substack{=-g^{\mu\rho}R_{\mu\rho}\\ =-R}}{}_{:\sigma}=0.\quad \square.$$

- raise suffix σ : $(R^{\sigma\alpha}-\frac{1}{2}g^{\alpha\sigma}R)_{:\alpha}=0$. pf: $2g^{\nu\sigma}R^\alpha{}_{\sigma:\alpha}-g^{\nu\sigma}R_{:\sigma}=0\ \underset{\sim}{\times\frac{1}{2}}\Rightarrow(R^{\nu\alpha}-g^{\nu\alpha}R)_{:\alpha}=0$.

- explicit form of Ricci tensor: $R_{\mu\nu}=\Gamma^\alpha{}_{\mu\alpha,\nu}-\Gamma^\alpha{}_{\mu\nu,\alpha}-\Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta}+\Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha}$.

- All terms are symmetrical in μ,ν.

▷ $g\equiv\det(g_{\mu\nu})$. $g_{,\nu}=\frac{\partial g}{\partial g_{\lambda\mu}}g_{\lambda\mu,\nu}=gg^{\lambda\mu}g_{\lambda\mu,\nu}$

for $\Gamma^\mu{}_{\nu\mu}$, $\Gamma^\mu{}_{\nu\mu}=g^{\lambda\mu}\Gamma_{\lambda\nu\mu}=\frac{1}{2}g^{\lambda\mu}(g_{\lambda\nu,\mu}+g_{\lambda\mu,\nu}-g_{\mu\nu,\lambda})$

$=\frac{1}{2}g^{\lambda\mu}g_{\lambda\mu,\nu}=\frac{1}{2}g^{-1}g_{,\nu}=\frac{1}{2}(\ln g)_{,\nu}$

$\Rightarrow\Gamma^\alpha{}_{\mu\alpha,\nu}=\frac{1}{2}(\ln g)_{,\mu\nu}=\frac{1}{2}(\ln g)_{,\nu\mu}=\Gamma^\alpha{}_{\nu\alpha,\mu}$.

7. Einstein's Law of gravitation

▷ Einstein's Assumption: in empty space, $\boxed{R_{\mu\nu}=0}$. empty: no matter present, no physical field except gravitational..

eg. the space between planets in the solar system is approximately empty.

$\begin{cases} \text{flat space: } R_{\mu\nu}=0 \text{ always holds. Geodensics are straight lines, so particle move along straight lines.} \\ \text{non-flat space: } R_{\mu\nu}=0 \text{ put restrictions on the curvature.} \end{cases}$

∘ looking upon $g_{\mu\nu}$ as potentials, then $R_{\mu\nu}=0$ appears as field eqns, nonlinear.

▷ Newtonian approximation

· consider a static gravitational field, refer to a static coordinate system

$$\Rightarrow g_{\mu\nu,0}=0 \; ; \; g_{m0}=0, \; g^{m0}=0, \; g^{00}=(g_{00})^{-1}. \quad (m=1,2,3).$$

$(g_{\mu\nu})=\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix}$

因为 ds^2 不会出现像 $2g_{10}dxdt$ 这样的项

▷ Roman suffixes like m and n take on the value 1, 2, 3.

$\Rightarrow \Gamma_{m0n}=0, \; \Gamma^{m}{}_{0n}=0.$ [With $\Gamma_{\mu\nu\sigma}=\frac{1}{2}(g_{\mu\nu,\sigma}+g_{\mu\sigma,\nu}-g_{\nu\sigma,\mu})$]

· consider a particle moving slowly. velocity in space $v^m \to 0$.

$\Rightarrow$ of the 1st order, $g_{00}v^{0^2}=1$ (natural unit)

the geodesic eqn: $\frac{dv^\sigma}{ds}+\Gamma^\sigma_{\mu\nu}v^\mu v^\nu=0 \xrightarrow{\text{becomes}} \frac{dv^m}{ds}\simeq-\Gamma^m{}_{00}v^{0^2}=-g^{mn}\Gamma_{n00}v^{0^2}=\frac{1}{2}g^{mn}g_{00,n}v^{0^2}$

Also, $\frac{dv^m}{ds}=\frac{dv^m}{dx^\mu}\frac{dx^\mu}{ds}\simeq\frac{dv^m}{dx^0}v^0 \Rightarrow \frac{dv^m}{dx^0}=\frac{1}{2}g^{mn}g_{00,n}v^0=g^{mn}(g_{00}^{\frac{1}{2}})_{,n} \xLeftrightarrow[\text{of } x^0]{g_{\mu\nu}\text{ indep.}} \underline{\frac{dv_m}{dx^0}=(g_{00}^{\frac{1}{2}})_{,m}}$

The particle move as if it were under a potential of $\sqrt{g_{00}}$

• with Einstein's law, suppose the gravitational field is weak, curvature small.

$\Rightarrow$ Choose a system s.t. curvature of the coordinate lines is small.

$\longrightarrow g_{\mu\nu}\simeq\text{const}, \; g_{\mu\nu,\sigma}\to 0, \; \Gamma_{\mu\nu\sigma}\to 0.$

$\Rightarrow$ of the 1st order, $R_{\mu\nu}\simeq\Gamma^\alpha{}_{\mu\alpha,\nu}-\Gamma^\alpha{}_{\mu\nu,\alpha}\simeq\frac{1}{2}g^{\rho\sigma}(g_{\rho\sigma,\mu\nu}-g_{\mu\rho,\nu\sigma}-g_{\nu\sigma,\rho\mu}+g_{\mu\nu,\rho\sigma})=0$

· take $\mu=\nu=0$, then $g^{mn}g_{00,mn}=0$

In this case, $\Box V=0 \longrightarrow g^{\mu\nu}V_{,\mu\nu}=0 \xrightarrow{\text{static}} g^{mn}V_{,mn}=0$ (laplace eqn) $\sim g_{00}$ satisfy the laplace eqn.

Choose unit of time s.t. $g_{00}\to 1$, when V small, we can let $g_{00}=1+2V$ (in this case, $g_{mn}\to -1$) (≠)

then $g_{00}^{\frac{1}{2}}\simeq(1+2V)^{\frac{1}{2}}\simeq 1+V$, then V becomes the potential, $\underbrace{\frac{dv^m}{dx^0}}_{\text{acceleration}}=\underbrace{g^{mn}}_{\simeq -1}\underbrace{(1+V)_{,n}}_{\text{grad } V}$ $\to$ just Newton's law

$\Longrightarrow$ Einstein's law of gravitation goes over Newton's law when the field is weak and static.

▷ the gravitational red shift. (同一光源在引力场不同处发射光的频率不同)

. consider a static gravitational field, and a static system.

An atom at rest emitting monochromatic radiation. $\Delta s \equiv$ the original, invariant peroid.

$\Rightarrow \Delta s^2=g_{00}\Delta x^{0^2}$, $\Delta x^0 \equiv$ local peroid. $\Rightarrow \Delta x^0 :: g_{00}^{-\frac{1}{2}} \sim \Delta x^0 :: 1-V.$

(static source, $ds^2=g_{00}dx^{0^2}$)

8 The Schwarzschild solution (Natural units, $c=1$, $G=1$)

▷ the sdn. of $R_{\mu\nu}=0$ in a static spherically symmetric field produced by a spherically symmetric body.

with a static system, $g_{\mu\nu,0}=0$, $g_{0m}=0$. Take $x^1=r$, $x^2=\theta$, $x^3=\phi$.

$\Rightarrow ds^2 = U dt^2 - V dr^2 - W r^2(d\theta^2+\sin^2\theta\, d\phi^2)$, U, V, W are func. of r only, for spherical symmetry

这是最方便的选择. 如果保留 $W(r)$, 无非是得到的 $g_{\mu\nu}$ 乘上一个 $W(r)$ 的系数.

$$\xrightarrow[U\to e^{2\nu},\ V\to e^{2\lambda}]{\text{let } W=1} ds^2 = \underbrace{e^{2\nu}}_{g_{00}} dt^2 \underbrace{-e^{2\lambda}}_{g_{11}} dr^2 \underbrace{-r^2}_{g_{22}} d\theta^2 \underbrace{-r^2\sin^2\theta}_{g_{33}} d\phi^2, \quad \nu=\nu(r),\ \lambda=\lambda(r).$$

$$\Rightarrow \Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu}+g_{\rho\nu,\mu}-g_{\mu\nu,\rho})$$

$\Gamma^1_{00} = \nu' e^{2\nu-2\lambda}$, $\Gamma^1_{11}=\lambda'$, $\Gamma^1_{22} = -re^{-2\lambda}$, $\Gamma^1_{33} = -r\sin^2\theta\, e^{-2\lambda}$.

$\Gamma^0_{10}=\Gamma^0_{01}=\nu'$. $\Gamma^2_{21}=\Gamma^2_{12}=\Gamma^3_{13}=\Gamma^3_{31}=r^{-1}$. $\Gamma^3_{23}=\Gamma^3_{32}=\cot\theta$. $\Gamma^2_{33} = -\sin\theta\cos\theta$.

$$\Rightarrow R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$$

$R_{00} = (-\nu''+\lambda'\nu'-\nu'^2-\frac{2\nu'}{r})e^{2\nu-2\lambda}$. $R_{11}=\nu''-\lambda'\nu'+\nu'^2-\frac{2\lambda'}{r}$. $R_{22}=(1+r\nu'-r\lambda')e^{-2\lambda}-1$. $R_{33}=R_{22}\sin^2\theta$.

By $R_{\mu\nu}=0$, R_{00} & $R_{11} \to \lambda'+\nu'=0$. for $r\to\infty$, space flat ($\lambda=\nu\to 0$) $\Rightarrow \lambda+\nu=0$.

$R_{22} \to (1+2r\nu')e^{2\nu}=1 \Leftrightarrow (re^{2\nu})' e^{2\nu}=1 \Rightarrow re^{2\nu}=r-2m$, $m\equiv$ const.

$\Rightarrow g_{00} = 1-\frac{2m}{r}$. Compare Newton Approximation: $g_{00}=1+2V \overset{V=-\frac{M}{r}}{=\!=\!=} 1-\frac{2M}{r}$. here const m is just the mass M of the central body.

$\Rightarrow ds^2 = (1-\frac{2m}{r})dt^2 - (1-\frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$ ~ the Schwarzchild sdn., holds ouside the surface of stars.

是牛顿引力理论的一个修正. 应用: 水星进动).

▷ Black holes.

in SI unit, $\begin{cases}[G]=[L]^3[T]^{-2}[M^{-1}] \\ [c]=[L][T]^{-1} \\ [r_s]=[L]\end{cases} \to r_s = 2m\frac{G}{c^2}$

- the sdn. above becomes singular at $r=2m$ ($g_{00}=0$, $g_{11}=-\infty$) $\Rightarrow$ $r_s=2m$. the Schwarzchild radius.
- consider a particle falling radially into the central body, velocity $v^\mu = \frac{dx^\mu}{ds}$. $\{t, r, \theta, \phi\} \longrightarrow v^2=v^3=0$.

By the geodesic eqn: $\frac{dv^0}{ds} = -\Gamma^0_{\mu\nu}v^\mu v^\nu = -g^{00}\Gamma_{0\mu\nu}v^\mu v^\nu = -g^{00}g_{00,1}v^0v^1 = -g^{00}\frac{dg_{00}}{dr}\frac{dr}{ds}v^0 = -g^{00}\frac{dg_{00}}{ds}v^0 = 0$.

$$\overset{g^{00}=1/g_{00}}{\Longrightarrow} g_{00}\frac{dv^0}{ds} + \frac{dg_{00}}{ds}v^0 = 0 \Rightarrow \frac{d(g_{00}v^0)}{ds} = 0 \Rightarrow g_{00}v^0 = k \equiv \text{const}.$$

↳ k is the value of g^{00} where the particle starts to fall. initially, $v^0=1$, $g_{00}|_{\text{initial}}=k$.

Also, $1 = g_{\mu\nu}v^\mu v^\nu = g_{00}{v^0}^2 + g_{11}{v^1}^2 \xrightarrow[g_{00}g_{11}=-1]{\times g_{00}} \underline{1-\frac{2m}{r} = g_{00} = k^2 - {v^1}^2}$

For falling body, $v^1<0$, $v^1 = -(k^2-1+\frac{2m}{r})^{\frac{1}{2}}$.

$$\Rightarrow \frac{dt}{dr} = \frac{v^0}{v^1} = -k(1-\frac{2m}{r})^{-1}(k^2-1+\frac{2m}{r})^{-\frac{1}{2}}.$$

▷ suppose $r=2m+\varepsilon$, $\varepsilon\to 0$. i.e. the particle is close to the critical radius.

$\frac{2m}{r} = \frac{1}{\frac{r}{2m}} = \frac{1}{1+\frac{\varepsilon}{2m}} \sim 1-\frac{\varepsilon}{2m}$

$$\Rightarrow \frac{dt}{dr} \sim -k(\tfrac{\varepsilon}{2m})^{-1}(k^2-\tfrac{\varepsilon}{2m})^{-\frac{1}{2}} \sim -k(\tfrac{\varepsilon}{2m})^{-1}\tfrac{1}{k} = -\frac{2m}{\varepsilon} = -\frac{2m}{r-2m}$$

$\Rightarrow t = -\int \frac{2m}{r-2m}dr = -2m\ln(r-2m)+C$. As $r\to 2m$, $t\to\infty$; $g_{00}^{-\frac{1}{2}} = (1-\frac{2m}{r})^{-\frac{1}{2}} \to \infty$.

All physical process on the particle will be observed to be going more and more slowly as $r\to 2m$.

▷ Consider an observer travelling with the particle. His time is measured by ds.

$\Rightarrow \frac{ds}{dr} = \frac{1}{v'} = -(k^2 - 1 + \frac{2m}{r})^{-\frac{1}{2}}$. As $r \to 2m$, $\frac{ds}{dr} \to -k^{-1}$, i.e., the observer reaches $r=2m$ in finite proper time.

▷ Afterwards? sailing through empty space into $r<2m$. ~ Analytic continuity

* use a nonstatic system of coordinates, $g_{\mu\nu}$ varying with time coordinate. $\{\tau \equiv t + f(r),\ \rho \equiv t + g(r),\ \theta,\ \phi\}$.

$(f' \equiv \frac{df}{dr})$. $d\tau^2 - \frac{2m}{r} d\rho^2 = (dt + f'dr)^2 - \frac{2m}{r}(dt + g'dr)^2$

choose f, g

$= (1 - \frac{2m}{r}) dt^2 + 2(f' - \frac{2m}{r} g') dr dt + (f'^2 - \frac{2m}{r} g'^2) dr^2$

s.t. $\begin{cases} f' = \frac{2m}{r} g' \\ \frac{2m}{r} g'^2 - f'^2 = (1 - \frac{2m}{r})^{-1} \end{cases}$

$= (1 - \frac{2m}{r}) dt^2 - (1 - \frac{2m}{r})^{-1} dr^2$

r=2m is a singularity here

* But once established the new coordinate system, we can disregard the previous one and the singularity no longer appears.

$\to g' = (\frac{r}{2m})^{\frac{1}{2}} (1 - \frac{2m}{r})^{-1} = \frac{dg}{dr} \Rightarrow g' - f' = (1 - \frac{2m}{r}) g' = (\frac{r}{2m})^{\frac{1}{2}} \xrightarrow{\int} g - f = \rho - \tau = \frac{2}{3} \frac{1}{\sqrt{2m}} r^{\frac{3}{2}} \Rightarrow \begin{cases} r = \mu(\rho - \tau)^{\frac{2}{3}} \\ \mu \equiv (\frac{3}{2}\sqrt{2m})^{\frac{2}{3}} \end{cases}$

$\Rightarrow g = \int (\frac{r}{2m})^{\frac{1}{2}} (1 - \frac{2m}{r})^{-1} dr \xrightarrow[2m=a^2]{r=y^2} \int \frac{2y^4}{a(y^2 - a^2)} dy = \frac{2}{3a} y^3 + 2ay - a^2 \ln \frac{y+a}{y-a}$

Substituting into Schwarzschild soln., $ds^2 = d\tau^2 - \frac{2m}{\mu(\rho-\tau)^{\frac{2}{3}}} d\rho^2 - \mu^2 (\rho - \tau)^{\frac{4}{3}} (d\theta^2 + \sin^2\theta\, d\phi^2)$. [soln.*]

$r = 2m \longleftrightarrow \rho - \tau = \frac{4m}{3}$. there is no singularity in ds^2 wrt $\{\tau, \rho, \theta, \phi\}$.

For ds^2 wrt $\{\tau, \rho, \theta, \phi\}$,

when $r > 2m$, it can be transform to the Schwarzschild soln. by a coordinate transf. $\Rightarrow$ it satisfies Einstein equation

$\Rightarrow$ when $r \leq 2m$ it also satisfies Einstein eqn. by analytic continuity (hold right down to $r=0$ or $\rho - \tau = 0$)

* the Schwarzchild soln. for empty space can be extended to $r<2m$. But ... $(r<2m) \cancel{\leftrightarrow} (r>2m)$ ~~communicate~~

9. Modification of the Einstein eqns by the presense of matter

· In empty space, $R^{\mu\nu} = 0 \Leftrightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0$ (LHS $\Rightarrow R=0 \Rightarrow$ RHS; RHS $\overset{缩并}{\Rightarrow} R=0 \Rightarrow$ LHS)

Introduce $X^{\mu\nu}, Y^{\mu\nu}$ to modify: $R^{\mu\nu} = X^{\mu\nu} \Leftrightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = Y^{\mu\nu} \longrightarrow Y^{\mu\nu}{}_{:\nu} = 0$

Bianci relation $(R^{\sigma\alpha} - \frac{1}{2} g^{\alpha\sigma} R)_{:\alpha} = 0$

in flat space $Y^{\mu\nu}{}_{,\nu} = 0$ conservation of energy & momentum

for convenience

$$\underline{\underline{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi Y^{\mu\nu}}}$$

in curved space it's only approximate conservation.

▷ the material energy tensor $\underline{Y^{\mu\nu} = ?}$

$Y^{\mu\nu}$ 是源的表征，可以是质量、能量场，也可以是电磁场等作为质场.

▷ the material energy tensor

- Consider a distribution of matter, velocity varies continuously.

Let z^μ denotes the coordinates of an element of the matter. $v^\mu \equiv \frac{dz^\mu}{ds}$ is a field func.

then $Y^{\mu\nu} = \underbrace{T^{\mu\nu} \equiv \rho v^\mu v^\nu}_{\text{the material energy tensor}}$. ρ is a scalar field s.t. ρv^μ determines the density and flow of the matter.

- 物理意义：$\rho v^0 \checkmark$ ~ 密度；$\rho v^m \checkmark$ 物质流（单位时间通过单位面积的质量 / 单位体积内的动量）

$\checkmark\ T^{00} = \rho v^0 v^0 \checkmark$：能量密度（c=1，质能等价）.

$\checkmark\ T^{0m} = \rho v^0 v^m \checkmark$：能流密度（单位时间通过单位面积的能量）

$\checkmark\ T^{n0} = \rho v^n v^0 \checkmark$：动量密度 ⇓（等价）

$\checkmark\ T^{nm} = \rho v^n v^m \checkmark$：动量流（单位时间通过单位面积的动量）

- conservation of matter: $(\rho v^\mu \checkmark)_{,\mu} = 0$. ←（欧氏空间中，$(\rho v^0)_{,0} = -(\rho v^m)_{,m}$，RHS为物质流的散度）.

or $(\rho v^\mu)_{:\mu} = 0$.

- verification: $T^{\mu\nu}{}_{:\nu} = 0$.

pf. $T^{\mu\nu}{}_{:\nu} = (\rho v^\mu v^\nu)_{:\nu} = v^\mu \underbrace{(\rho v^\nu)_{:\nu}}_{\substack{\text{conservation of matter} \\ =0}} + \rho \underbrace{v^\nu v^\mu{}_{:\nu}}_{=0,\ \text{if geodesics} \ldots}$.

- If matter moves along geodesics, then $\frac{dv^\mu}{ds} + \Gamma^\mu_{\nu\sigma} v^\nu v^\sigma = 0 \Leftrightarrow (v^\mu{}_{,\nu} + \Gamma^\mu_{\nu\sigma} v^\sigma) v^\nu = 0 \Leftrightarrow \underline{v^\mu{}_{:\nu} v^\nu = 0}$. ↙ 测地线方程的等价形式

- How ρ varies along the world line of an element of matter?

with condition of conservation of mass $(\rho v^\mu)_{:\mu} = 0$, we have $\rho_{:\mu} v^\mu + \rho v^\mu{}_{:\mu} = \rho_{,\mu} v^\mu + \rho v^\mu{}_{:\mu} = 0$.

↑ ρ scalar.

$\Rightarrow \frac{d\rho}{ds} = \rho_{,\mu} v^\mu = -\rho v^\mu{}_{:\mu}$. ?

? 上式是决定 ρ 沿物质元的世界线如何变化的条件. 它允许 ρ 任意地从一物质元的世界线改变到相邻物质元的世界线. 因而我们可以规定在时空中成管状的一束世界线外，ρ 都等于零. 这样一束世界线就组成一个有限尺寸的物质粒子，在粒子外，我们有 $\rho = 0$，真空的爱因斯坦方程成立.

ps: $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi \rho v^\mu v^\nu$ 成立，

可推出质量守恒、物质元沿世界线运动.

10. Tensor calculus

(Dirac GR特点：引出为主、从特例得出规律，但并非证明.)

▷ Tensor densities

in coordinates transf. in dim-4 space, an element:

$$dx^{0'}dx^{1'}dx^{2'}dx^{3'} = J\, dx^0 dx^1 dx^2 dx^3 \Leftrightarrow d^4x' = J d^4x, \quad J = \frac{\partial(x^{0'}x^{1'}x^{2'}x^{3'})}{\partial(x^0x^1x^2x^3)} = \det(x^{\mu'}{}_{,\sigma})$$

for $g_{\alpha\beta}$, $g_{\alpha\beta} = x^{\mu'}{}_{,\alpha}\, g_{\mu'\nu'}\, x^{\nu'}{}_{,\beta} \xrightarrow{\det} g = J^2 g'$, $g \equiv \det(g_{\alpha\beta})$.

since $g<0$, then $\sqrt{-g} = J\sqrt{-g'}$. [比如 $\det\eta = \det(1, -1, -1, -1) = -1 < 0$].

For S is a scalar field quantity, $S = S'$, then $\int_\Omega S\sqrt{-g}\, d^4x = \int_\Omega S\sqrt{-g'}\, J\, d^4x = \int_{\Omega'} S'\sqrt{-g'}\, d^4x'$ ~ invariant

⇒ Def $S\sqrt{-g}$ as a <u>scalar density</u>, meaning a quantity whose integral is invariant.

Similarly, for any tensor field $T^{\mu\nu\cdots}$, we call $T^{\mu\nu\cdots}\sqrt{-g}$ a tensor density.

* $\int_\Omega T^{\mu\nu}\sqrt{-g}\, d^4x$ is approximately a tensor if Ω small, but not a tensor if Ω large. 若Ω很大，则积分由不同点的张量之和组成，进而在坐标变换下无法满足变换律.

- some eqns in the form of $\sqrt{-g} \equiv \surd$. [√可以理解为空间属性的表征]

for $2\surd^{-1}\surd_{,\nu} = 2(-g)^{-\frac{1}{2}}\frac{\partial(-g)^{\frac{1}{2}}}{\partial x^\nu} = g^{-1}g_{,\nu}$,

$$g_{,\nu} = g g^{\lambda\mu} g_{\lambda\mu,\nu} \Leftrightarrow \surd_{,\nu} = \frac{1}{2}\surd\, g^{\lambda\mu}g_{\lambda\mu,\nu}$$

$$\Gamma^\mu{}_{\nu\mu} = \frac{1}{2}g^{-1}g_{,\nu} \Leftrightarrow \Gamma^\mu{}_{\nu\mu}\surd = \surd_{,\nu}$$

▷ Gauss and stokes thm

For a contravariant vector A^μ, its <u>covariant divergence</u> is $A^\mu{}_{:\mu}$, a scalar.

- $A^\mu{}_{:\mu} = A^\mu{}_{,\mu} + \Gamma^\mu{}_{\nu\mu}A^\nu = A^\mu{}_{,\mu} + \surd^{-1}\surd_{,\nu}A^\nu \Rightarrow A^\mu{}_{:\mu}\surd = (A^\mu\surd)_{,\mu}$

for $\int S\sqrt{-g}\, d^4x$ = invariant, $S \to A^\mu{}_{:\mu} \Rightarrow \int A^\mu{}_{:\mu}\surd\, d^4x = \int (A^\mu\surd)_{,\mu}\, d^4x \equiv$ const.

- if $A^\mu{}_{:\mu} = 0$, then $(A^\mu\surd)_{,\mu} = 0 \Rightarrow \underline{(A^0\surd)_{,0} = -(A^m\surd)_{,m}} \Rightarrow \underline{\left(\int A^0\surd\, dx^3\right)_{,0} = -\int (A^m\surd)_{,m}\, d^3x}$ ~ A conservation law.

for a fluid, density $A^0\surd$, flow $A^m\surd$, then it's the conservation of a fluid. e.g. $\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{j} = -j^m{}_{,m}$ ($\sqrt{-g}=1$)

By Gauss's thm, $\int_\Omega B^\mu{}_{,\mu}\, d^3x$ can be written as a surface int. over $\partial\Omega$.

- In flat space, $\int Y^{\mu\nu}{}_{,\nu}\, d^4x$ can be expressed as a surface int.; But in a curved space, $\int Y^{\mu\nu}{}_{:\nu}\surd\, d^4x$ cannot.

- Exception: when $F^{\mu\nu} = -F^{\nu\mu}$. (e.g. electromagnetic field)

$$F^{\mu\nu}{}_{:\sigma} = F^{\mu\nu}{}_{,\sigma} + \Gamma^\mu{}_{\sigma\rho}F^{\rho\nu} + \Gamma^\nu{}_{\sigma\rho}F^{\mu\rho}$$

$$\Rightarrow F^{\mu\nu}{}_{:\nu} = F^{\mu\nu}{}_{,\nu} + \underbrace{\Gamma^\mu{}_{(\nu\rho)}F^{[\rho\nu]}}_{=0} + \underbrace{\Gamma^\nu{}_{\nu\rho}}_{\surd^{-1}\surd_{,\rho}}F^{\mu\rho} = F^{\mu\nu}{}_{,\nu} + \surd^{-1}\surd_{,\rho}F^{\mu\rho}$$

$$\Rightarrow F^{\mu\nu}{}_{:\nu}\surd = (F^{\mu\nu}\surd)_{,\nu}$$

i.e. $\int F^{\mu\nu}{}_{:\nu}\surd\, d^4x$ = a surface int.; $F^{\mu\nu}{}_{:\nu} = 0 \to$ conservation law.

- in the case $Y^{\mu\nu} = Y^{\nu\mu}$.

$$Y_\mu{}^\nu{}_{:\sigma} = Y_\mu{}^\nu{}_{,\sigma} - \Gamma^\alpha{}_{\mu\sigma}Y_\alpha{}^\nu + \Gamma^\nu{}_{\alpha\sigma}Y_\mu{}^\alpha \xrightarrow{\sigma=\nu} Y_\mu{}^\nu{}_{:\nu} = Y_\mu{}^\nu{}_{,\nu} + \surd^{-1}\surd_{,\alpha}Y_\mu{}^\alpha - \underline{\Gamma_{\alpha\mu\nu}Y^{\alpha\nu}} = Y_\mu{}^\nu{}_{,\nu} + \surd^{-1}\surd_{,\alpha}Y_\mu{}^\alpha - \frac{1}{2}g_{\alpha\nu,\mu}Y^{\alpha\mu}$$

$$= \frac{1}{2}(\Gamma_{\alpha\mu\nu} + \Gamma_{\nu\mu\alpha})Y^{\alpha\nu}$$
$$= \frac{1}{2}(\Gamma_{\alpha\nu\mu} + \Gamma_{\nu\alpha\mu})Y^{\alpha\nu}$$
$$= \frac{1}{2}\Gamma_{\alpha\nu,\mu}Y^{\alpha\nu}$$

$$\Rightarrow Y_\mu{}^\nu{}_{:\nu}\surd = (Y_\mu{}^\nu\surd)_{,\nu} - \frac{1}{2}g_{\alpha\beta,\mu}Y^{\alpha\beta}$$

- $\frac{1}{2}\int_{\partial\Omega}(A_{\mu:\nu} - A_{\nu:\mu})\, dS^{\mu\nu} = \int_\Omega A_\mu\, dx^\mu$, $dS^{\mu\nu} = dx^{[\mu}dx^{\nu)}$. (细节涉及微分形式).

▷ Harmonic coordinates : under which cons. do x^λ 's satisfy $\Box x^\lambda = 0$?

Condition : $g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$.

$\Longleftrightarrow (g^{\mu\nu}\sqrt{\ })_{,\nu} = 0$.

[for $-g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu}\sqrt{\ } = (g^{\lambda\nu}\sqrt{\ })_{,\nu}$]

explanation: $\Box V = 0 \Leftrightarrow g^{\mu\nu}(V_{,\mu\nu} - \Gamma^\alpha{}_{\mu\nu} V_{,\alpha}) = 0 \xrightarrow[x^\lambda{}_{,\alpha} = \delta^\lambda_\alpha]{V \to x^\lambda} g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$.

$\because g^{\mu\nu}{}_{,\sigma} = -g^{\mu\alpha}g^{\nu\beta}(\Gamma_{\alpha\beta\sigma} + \Gamma_{\beta\alpha\sigma}) = -g^{\nu\beta}\Gamma^\mu{}_{\beta\sigma} - g^{\mu\alpha}\Gamma^\nu{}_{\alpha\sigma}$.

$\therefore (g^{\mu\nu}\sqrt{\ })_{,\sigma} = g^{\mu\nu}{}_{,\sigma}\sqrt{\ } + g^{\mu\nu}\sqrt{\ }_{,\sigma} = (-g^{\nu\beta}\Gamma^\mu{}_{\beta\sigma} - g^{\mu\alpha}\Gamma^\nu{}_{\alpha\sigma} + g^{\mu\nu}\Gamma^\beta{}_{\sigma\beta})\sqrt{\ }$

$\Gamma^\mu{}_{\nu\mu}\sqrt{\ } = \sqrt{\ }_{,\nu}$

$\sigma = \nu$

$(g^{\mu\nu}\sqrt{\ })_{,\nu} = -g^{\nu\beta}\Gamma^\mu{}_{\beta\nu}\sqrt{\ }$. □.

?

▷ e.g. the electromagnetic field

Maxwell's eqns :
$$\begin{cases} E = -\frac{1}{c}\frac{\partial A}{\partial t} - \text{grad}\,\phi \\ H = \text{curl}\,A \end{cases}$$

(CGSM unit) ?

$$\begin{cases} \frac{1}{c}\frac{\partial H}{\partial t} = -\text{curl}\,E \\ \text{div}\,H = 0 \\ \frac{1}{c}\frac{\partial E}{\partial t} = \text{curl}\,H - \frac{4\pi}{c} j \\ \text{div}\,E = 4\pi\rho \end{cases}$$

$$\begin{cases} [F_{\mu\nu}] = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{bmatrix} \\ [F^{\mu\nu}] = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & H^3 & -H^2 \\ E^2 & -H^3 & 0 & H^1 \\ E^3 & H^2 & -H^1 & 0 \end{bmatrix} \end{cases}$$

The potentials A and ϕ form a 4-vector κ^μ : $\kappa^0 = \phi$, $\kappa^m = A^m$.

Def $F_{\mu\nu} = \kappa_{\mu,\nu} - \kappa_{\nu,\mu}$ ~ the electromagnetic tensor

$\Rightarrow E^1 = -\frac{\partial \kappa^1}{\partial x^0} - \frac{\partial \kappa^0}{\partial x^1} = \frac{\partial \kappa_1}{\partial x^0} - \frac{\partial \kappa_0}{\partial x^1} = F_{10} = -F^{10}$; $H^1 = \frac{\partial \kappa^3}{\partial x^2} - \frac{\partial \kappa^2}{\partial x^3} = -\frac{\partial \kappa_3}{\partial x^2} + \frac{\partial \kappa_2}{\partial x^3} = F_{23} = F^{23}$ . . . .

$\eta = (1, -1, -1, -1)$

· By def. $F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0 \Longrightarrow \begin{cases} \text{curl}\,E = \frac{1}{c}\frac{\partial H}{\partial t} \\ \text{div}\,H = 0 \end{cases}$

$$\left.\begin{aligned} &F^{0\nu}{}_{,\nu} = F^{0m}{}_{,m} = -F^{m0}{}_{,m} = \text{div}\,E = 4\pi\rho \\ &F^{1\nu}{}_{,\nu} = F^{10}{}_{,0} + F^{12}{}_{,2} + F^{13}{}_{,3} = -\frac{\partial E^1}{\partial x^0} + \frac{\partial H^3}{\partial x^2} - \frac{\partial H^2}{\partial x^3} = 4\pi j^1 \end{aligned}\right\} F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu .$$

def J^μ s.t. $J^0 = \rho$ the charge density and $J^m = j^m$.

▷ invariant form.

· $F_{\mu\nu} = \kappa_{\mu,\nu} - \kappa_{\nu,\mu} = \kappa_{\mu:\nu} - \kappa_{\nu:\mu}$.

Maxwell's eqn
$$\begin{cases} F_{\mu\nu:\sigma} = F_{\mu\nu,\sigma} - \Gamma^\alpha{}_{\mu\sigma} F_{\alpha\nu} - \Gamma^\alpha{}_{\nu\sigma} F_{\mu\alpha} \xrightarrow[\text{add up}]{\text{cyclic } \mu,\nu,\sigma} F_{\mu\nu:\sigma} + F_{\nu\sigma:\mu} + F_{\sigma\mu:\nu} = F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0. \\ F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu \xrightarrow{?} F^{\mu\nu}{}_{:\nu} = 4\pi J^\mu \end{cases}$$

[解释: 在闵氏时空中, 直线坐标下, $F^{\mu\nu}{}_{,\nu} = F^{\mu\nu}{}_{:\nu}$. 其他时空/坐标下, $F^{\mu\nu}{}_{,\nu} \to F^{\mu\nu}{}_{:\nu}$ 是一种推广. 本书未解释此举的可行性.]

Collary: $(F^{\mu\nu}\sqrt{\ })_{,\nu} = F^{\mu\nu}{}_{:\nu}\sqrt{\ } = 4\pi J^\mu\sqrt{\ } \Rightarrow (J^\mu\sqrt{\ })_{,\nu} = (4\pi)^{-1}(F^{\mu\nu}\sqrt{\ })_{,\mu\nu} = 0$. it's the conservation law.

11. The gravitational action principle

▷ The action without matter, i.e. the gravity part of the action, I_g

- Introduce $I \equiv \int R \surd \, d^4x$. $\delta I = 0$ wrt $\delta g_{\mu\nu}$ gives Einstein's vacuum eqns. ($\delta g_{\mu\nu} = 0$ on the boundary)

write $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} (\Gamma^\alpha{}_{\mu\alpha,\nu} - \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta} + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha})$

$\equiv R^* - L$. where $R^* \equiv g^{\mu\nu}(\Gamma^\sigma{}_{\mu\sigma,\nu} - \Gamma^\sigma{}_{\mu\nu,\sigma})$, $L \equiv g^{\mu\nu}(\Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho})$

int. by part, $R^*\surd = (g^{\mu\nu}\Gamma^\sigma{}_{\mu\sigma}\surd)_{,\nu} - (g^{\mu\nu}\Gamma^\sigma{}_{\mu\nu}\surd)_{,\sigma} - (g^{\mu\nu}\surd)_{,\nu}\Gamma^\sigma{}_{\mu\sigma} + (g^{\mu\nu}\surd)_{,\sigma}\Gamma^\sigma{}_{\mu\nu}$.

$\begin{cases} (g^{\mu\nu}\surd)_{,\sigma} = (-2g^{\nu\beta}\Gamma^\mu{}_{\sigma\beta} + g^{\mu\nu}\Gamma^\rho{}_{\sigma\rho})\surd \\ (g^{\mu\nu}\surd)_{,\nu} = -g^{\nu\rho}\Gamma^\mu{}_{\rho\nu}\surd \end{cases}$

$\hookrightarrow \simeq g^{\nu\beta}\Gamma^\mu{}_{\beta\nu}\Gamma^\sigma{}_{\mu\sigma} + (-2g^{\nu\beta}\Gamma^\mu{}_{\beta\sigma} + g^{\mu\nu}\Gamma^\beta{}_{\sigma\beta})\Gamma^\sigma{}_{\mu\nu}\surd = 2L\surd$

def $\mathcal{L} \equiv L\surd$, then $I \simeq \int L\surd\, d^4x = \int \mathcal{L}\, d^4x$

还能不是的吗？
(ps: 因为扔了2个表面项，故 $\mathcal{L}$ 不是张量密度，而$R\surd$是张量密度)

▷ $I = \int \mathcal{L}\, d^4x = \int dx^0 \int \mathcal{L}\, dx^1 dx^2 dx^3$

the Lagrangian density in 3D / the action density in 4D

the action

the largrangian

then $\delta I = \int dx^0 \int \delta\mathcal{L}\, dx^1 dx^2 dx^3$.

$$\delta\mathcal{L} = \delta(\Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta}g^{\mu\nu}\surd) - \delta(\Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}g^{\mu\nu}\surd)$$

$\begin{cases} (g^{\mu\nu}\surd)_{,\nu} = -g^{\nu\beta}\Gamma^\mu{}_{\beta\nu}\surd \\ \Gamma^\mu{}_{\nu\mu}\surd = \surd_{,\nu} \\ g^{\mu\nu}{}_{,\sigma} = -g^{\nu\alpha}\Gamma^\mu{}_{\alpha\sigma} - g^{\mu\alpha}\Gamma^\nu{}_{\alpha\sigma} \end{cases}$

$$= \{\Gamma^\alpha{}_{\mu\nu}\delta(\Gamma^\beta{}_{\alpha\beta}g^{\mu\nu}\surd) + \Gamma^\beta{}_{\alpha\beta}g^{\mu\nu}\surd\,\delta\Gamma^\alpha{}_{\mu\nu}\} - \{2(\delta\Gamma^\beta{}_{\mu\alpha})\Gamma^\alpha{}_{\nu\beta}g^{\mu\nu}\surd + \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}\delta(g^{\mu\nu}\surd)\}$$

$$= \{\Gamma^\alpha{}_{\mu\nu}\delta(g^{\mu\nu}\surd_{,\alpha}) + \Gamma^\beta{}_{\alpha\beta}\delta(g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu}\surd) - \Gamma^\beta{}_{\alpha\beta}\Gamma^\alpha{}_{\mu\nu}\delta(g^{\mu\nu}\surd)\} - \{2\delta(\Gamma^\beta{}_{\mu\alpha}g^{\mu\nu}\surd)\Gamma^\alpha{}_{\nu\beta} - \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}\delta(g^{\mu\nu}\surd)\}$$

$\mu \leftrightarrow \beta$

$$= \Gamma^\alpha{}_{\mu\nu}\delta(g^{\mu\nu}\surd_{,\alpha}) - \Gamma^\beta{}_{\alpha\beta}\delta(g^{\alpha\nu}\surd)_{,\nu} + \delta(g^{\nu\beta}{}_{,\alpha}\surd)\Gamma^\alpha{}_{\nu\beta} + (\Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta} - \Gamma^\beta{}_{\alpha\beta}\Gamma^\alpha{}_{\mu\nu})\delta(g^{\mu\nu}\surd)$$

$$\simeq (-\Gamma^\alpha{}_{\mu\nu,\alpha} + \Gamma^\beta{}_{\alpha\beta,\nu} + \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta} - \Gamma^\beta{}_{\alpha\beta}\Gamma^\alpha{}_{\mu\nu})\delta(g^{\mu\nu}\surd)$$

$$= R_{\mu\nu}\delta(g^{\mu\nu}\surd)$$

$$\Rightarrow \delta I = \delta\int\mathcal{L}\, d^4x = \int R_{\mu\nu}\delta(g^{\mu\nu}\surd)\, d^4x \xrightarrow{\delta I = 0} R_{\mu\nu} = 0. \quad \square. \text{ (form 1)}$$

- $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$. Pf: $g^{\alpha\mu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\alpha\mu} = \delta(g_\nu{}^\alpha) = 0 \xrightarrow{\times g^{\beta\nu}} g_{\mu\nu}g^{\beta\nu}\delta g^{\alpha\mu} = \delta g^{\alpha\beta} = -g^{\beta\nu}g^{\alpha\mu}\delta g_{\mu\nu}$.
- $\delta\surd = \frac{1}{2}\surd g^{\alpha\beta}\delta g_{\alpha\beta}$. pf: $\delta\surd = \delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial(-g)}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\surd^{-1}(-g)g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\surd g^{\mu\nu}\delta g_{\mu\nu}$.

$$\Rightarrow \delta(g^{\mu\nu}\surd) = \surd\delta g^{\mu\nu} + g^{\mu\nu}\delta\surd = -(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu})\surd\delta g_{\alpha\beta}.$$

$$\Rightarrow \delta I = -\int R_{\mu\nu}(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu})\surd\delta g_{\alpha\beta}\, d^4x$$

$$= -\int (R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R)\surd\delta g_{\alpha\beta}\, d^4x \xrightarrow[\delta I = 0]{} R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = 0. \quad \square. \text{ (form 2)}$$

▷ the action for a continuous distribution of matter. $I = I_g + I_m$. 　　I_g 为 $\tilde{I}_g$ 乘上系数.

$$\delta I = \delta(I_g + I_m) = 0 \Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi \rho v^\mu v^\nu \equiv -8\pi T^{\mu\nu}.$$

▷ $I_m = -\int \underset{\substack{\uparrow \\ \text{density}}}{\rho} \sqrt{\ }\, d^4x \underset{\uparrow}{=} -\int (p^\mu p_\mu)^{\frac{1}{2}} d^4x$

$p^\mu \equiv \rho v^\mu \sqrt{\ }$;
$(p^\mu p_\mu)^{\frac{1}{2}} = \rho \sqrt{\ }$

◦ 构造动机: 令 $p^\mu \equiv (p, p^1, p^2, p^3)\sqrt{\ } = \rho v^\mu \sqrt{\ }$ 表征物质场某点处的密度与物质流.

则 $p^0 dx^1 dx^2 dx^3 \sim$ 体元 $dx^1 dx^2 dx^3$ 内的质量; $p^1 dx^0 dx^2 dx^3 \sim$ 在时间间隔 dx^0 内流过面元 $dx^2 dx^3$ 的物质.

假设物质守恒, i.e. $p^\mu{}_{,\mu} = 0$.　　b^0: 时间变化; b^m: 质元位移.

考虑一区域 Ω, Ω 中质元坐标为 z^μ, 变化到 $z^\mu + \delta z^\mu$, $\delta z^\mu \equiv b^\mu$, 这将引起 Ω 内质量的变化.

使用积分形式, $\delta \int_\Omega p^0 dx^1 dx^2 dx^3 = -\int_{\partial\Omega} p^0 b^r dS_r \underset{?}{+} \int_{\partial\Omega} p^r b^0 dS_r$

区域 Ω 3-体积内质量的变化量.　　质元位移导致的变化 (∵与时间)　　时间演化导致的物质流动

$\xrightarrow{\text{Gauss thm}} \delta p^0 = (p^r b^0 - p^0 b^r)_{,r} \xrightarrow{\text{generalize}} \delta p^\mu = (p^\nu b^\mu - p^\mu b^\nu)_{,\nu}$

自洽性检验: $p^\mu \propto b^\mu$ 时, 质元沿世界线变化, $\delta p^0 = 0$, 成立.

对于点粒子, 自然取 $-m\int ds$ 为作用量. 比较, 将 $m \to p^0 dx^1 dx^2 dx^3$ 即可.

于是取 $I_m = -\int p^0 ds\, dx^1 dx^2 dx^3 = -\int \rho \sqrt{\ }\, v^0 ds\, dx^1 dx^2 dx^3 = -\int \rho \sqrt{\ }\, d^4x$.

◦ Variation: $I_m = -\int \rho \sqrt{\ }\, d^4x = -\int (g_{\mu\nu} p^\mu p^\nu \sqrt{\ })\, d^4x$.　　$[\, p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = \rho^2 v^\mu v_\mu \sqrt{\ }^2 = (\rho\sqrt{\ })^2 \,]$

$$\delta (g_{\mu\nu} p^\mu p^\nu)^{\frac{1}{2}} = \frac{1}{2}(p^\lambda p_\lambda)^{-\frac{1}{2}} (p^\mu p^\nu \delta g_{\mu\nu} + 2 p_\mu \delta p^\mu)$$
$$= \frac{1}{2} \rho v^\mu v^\nu \sqrt{\ }\, \delta g_{\mu\nu} + v_\mu \delta p^\mu.$$

$$\int v_\mu \delta p^\mu d^4x = \int v_\mu (p^\nu b^\mu - p^\mu b^\nu)_{,\nu} d^4x$$

$\delta p^\mu = (p^\nu b^\mu - p^\mu b^\nu)_{,\nu}$

$$= \int v_{\mu,\nu} (p^\nu b^\mu - p^\mu b^\nu) d^4x$$
$$= \int (v_{\mu,\nu} - v_{\nu,\mu}) p^\nu b^\mu d^4x$$

$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ ↳
$$= \int (v_{\mu;\nu} - v_{\nu;\mu}) \rho v^\nu b^\mu \sqrt{\ }\, d^4x.$$

$g_{\mu\nu} v^\mu v^\nu = 1 \to v^\nu v_{\nu;\mu} = 0$ ↳
$$= \int v_{\mu;\nu} v^\nu \rho b^\mu \sqrt{\ }\, d^4x.$$

$$\Rightarrow \delta I = \delta\left(\frac{1}{16\pi} I_g + I_m\right)$$
$$= -\int \underbrace{(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 8\pi \rho v^\mu v^\nu)}_{\text{Einstein field eqn}} \frac{1}{16\pi} \sqrt{\ }\, \delta g_{\mu\nu} d^4x - \int \underbrace{v_{\mu;\nu} v^\nu}_{\text{geodesic eqn}} \rho b^\mu \sqrt{\ }\, d^4x$$

▷ the action for the electromagnetic field.

- $\mathcal{L}_{em} \equiv (8\pi)^{-1}(E^2 - H^2) = -(16\pi)^{-1} F_{\mu\nu}F^{\mu\nu}$. (展开验证. 显然的).

  In GR, $I_{em} = -(16\pi)^{-1}\int F_{\mu\nu}F^{\mu\nu}\sqrt{\ }\, d^4x$.

Recall: $F_{\mu\nu} = K_{\mu:\nu} - K_{\nu:\mu}$. $K^{\mu} = (\phi, A^1, A^2, A^3)$.

① varing $g_{\mu\nu}$: $\delta(F_{\mu\nu}F^{\mu\nu}\sqrt{\ }) = \delta(F_{\mu\nu}F_{\alpha\beta}\, g^{\mu\alpha}g^{\nu\beta}\sqrt{\ })$

$$= F_{\mu\nu}F^{\mu\nu}\delta\sqrt{\ } + F_{\mu\nu}F_{\alpha\beta}\sqrt{\ }\,\delta(g^{\mu\alpha}g^{\nu\beta})$$

$\to -F_{\mu\nu}F_{\alpha\beta}\, g^{\mu\alpha}g^{\nu\sigma}g^{\beta\rho}\delta g_{\rho\sigma} - F_{\mu\nu}F_{\alpha\beta}\, g^{\nu\beta}g^{\mu\sigma}g^{\alpha\rho}\delta g_{\rho\sigma}$

$\mu \leftrightarrow \nu \Downarrow \alpha \leftrightarrow \beta$

$-F_{\mu\nu}F_{\alpha\beta}\, g^{\mu\alpha}g^{\nu\sigma}g^{\beta\rho}\delta g_{\rho\sigma}$

$$\begin{cases} \delta\sqrt{\ } = \frac{1}{2}\sqrt{\ }\, g^{\alpha\beta}\delta g_{\alpha\beta} \\ \delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} \end{cases}$$

$$= \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\sqrt{\ }\, g^{\rho\sigma}\delta g_{\rho\sigma} - 2F_{\mu\nu}F_{\alpha\beta}\, g^{\mu\alpha}g^{\nu\sigma}g^{\beta\rho}\sqrt{\ }\,\delta g_{\rho\sigma}$$

$$= \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}g^{\rho\sigma} - 2F^{\nu\sigma}F_{\nu}{}^{\rho}\right)\sqrt{\ }\,\delta g_{\rho\sigma}$$

$$\equiv 8\pi E^{\rho\sigma}\sqrt{\ }\,\delta g_{\rho\sigma}$$

▷ $4\pi E^{\rho\sigma} \equiv -F^{\rho}{}_{\nu}F^{\sigma\nu} + \frac{1}{4}g^{\rho\sigma}F_{\mu\nu}F^{\mu\nu}$. $E^{\rho\sigma}$ is the stress-energy tensor of the electromagnetic field. $E^{\rho\sigma} = E^{\sigma\rho}$.

$F^{\rho}{}_{\nu}F^{\sigma\nu} = g_{\mu\nu}F^{\rho\mu}F^{\sigma\nu}$. symmetrical

in special relativity, $g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$.

$$\begin{cases} 4\pi E^{00} = g_{\mu\nu}F^{0\mu}F^{0\nu} + \frac{1}{4}g^{00}F_{\mu\nu}F^{\mu\nu} = E^2 + \frac{1}{4}(-2E^2 + 2H^2) = \frac{1}{2}(E^2 + H^2). \\ 4\pi E^{01} = g_{\mu\nu}F^{0\mu}F^{1\nu} + \frac{1}{4}g^{01}F_{\mu\nu}F^{\mu\nu} = -F^{02}F^{12} - F^{03}F^{13} = E^2H^3 - E^3H^2 = (\vec{E}\times\vec{H})_1. \end{cases}$$

$E^{00} \sim$ the energy density.

E^{0n}: the Poynting vector / the rate of flow of energy.

② varing K_μ. $\delta(F_{\mu\nu}F^{\mu\nu}\sqrt{\ }) = 2F^{\mu\nu}\sqrt{\ }\,\delta F_{\mu\nu} = 2F^{\mu\nu}\sqrt{\ }\,\delta(K_{\mu,\nu} - K_{\nu,\mu}) = 4F^{\mu\nu}\sqrt{\ }\,\delta K_{\mu,\nu}$

$$= 4(F^{\mu\nu}\sqrt{\ }\,\delta K_\mu)_{,\nu} - 4(F^{\mu\nu}\sqrt{\ })_{,\nu}\delta K_\mu$$

$$= 4(F^{\mu\nu}\sqrt{\ }\,\delta K_\mu)_{,\nu} - 4F^{\mu\nu}{}_{:\nu}\sqrt{\ }\,\delta K_\mu$$

$F^{\mu\nu}{}_{:\nu}\sqrt{\ } = (F^{\mu\nu}\sqrt{\ })_{,\nu}$ if $F^{\mu\nu} = -F^{\nu\mu}$.

$$\Rightarrow \delta I_{em} = \int \left[-\frac{1}{2}E^{\rho\sigma}\delta g_{\mu\nu} + (4\pi)^{-1}F^{\mu\nu}{}_{:\nu}\delta K_\mu\right]\sqrt{\ }\, d^4x.$$

▷ the action of charged matter

- For a single particle of charge e, the action: $-e\int_L K_\mu dx^\mu = -e\int_L K_\mu v^\mu ds$. L is the world line.
- Introduce $\mathfrak{J}^\mu$ to determine the density and flow of electricity.

  Similarly, $\delta\mathfrak{J}^\mu = (\mathfrak{J}^\nu b^\mu - \mathfrak{J}^\mu b^\nu)_{,\nu}$, with the charged matter moves from z^μ to $z^\mu + b^\mu$.

$\therefore$ def $I_q \equiv -\int \mathfrak{J}^0 K_\mu v^\mu dx^1dx^2dx^3ds$, $\mathfrak{J}^\mu = \sigma v^\mu\sqrt{\ }$, σ is the charged density.

$$\Rightarrow \quad I_q = -\int \sigma K_\mu v^\mu\sqrt{\ }\, d^4x = -\int K_\mu\mathfrak{J}^\mu d^4x.$$

- variation: $\delta I_q = -\int [\mathfrak{J}^\mu\delta K_\mu + K_\mu(\mathfrak{J}^\nu b^\mu - \mathfrak{J}^\mu b^\nu)_{,\nu}]\, d^4x$

$$\simeq \int [-\sigma v^\mu\sqrt{\ }\,\delta K_\mu + K_{\mu,\nu}(\mathfrak{J}^\nu b^\mu - \mathfrak{J}^\mu b^\nu)]\, d^4x$$

$$= \int [-\sigma v^\mu\sqrt{\ }\,\delta K_\mu + F_{\mu\nu}\mathfrak{J}^\nu b^\mu]\, d^4x$$

$$= \int [-v^\mu\delta K_\mu + F_{\mu\nu}v^\nu b^\mu]\,\sigma\sqrt{\ }\, d^4x.$$

▷ $\delta(\tilde{I}_g + I_m + I_{em} + I_q) = 0$.

$$\hookrightarrow \begin{cases} R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi\rho v^\mu v^\nu + 8\pi E^{\mu\nu} = 0 \quad ① & (\cdots \sqrt{}\,\delta g_{\mu\nu}) \\ -\sigma v^\mu + (4\pi)^{-1} F_{\mu\nu:\nu} = 0 \quad ② & (\cdots \sqrt{}\,\delta K_\mu) \\ \rho v_{\mu:\nu} v^\nu + \underbrace{F_{\mu\nu}\sigma v^\nu}_{} = 0 \quad ③ & (\cdots \sqrt{}\, b^\mu) \end{cases}$$

② is one of the Maxwell's eqn.

$= F_{\mu\nu} J^\nu$, the Lorentz force, causing the trajectory of an element of matter to depart from a geodesic.

◦ ①, ② $\Rightarrow$ ③. i.e., not indep.

$$\nabla_\nu ① = (\rho v^\mu v^\nu + E^{\mu\nu})_{:\nu} = 0. \Rightarrow 4\pi E^{\mu\nu}{}_{:\nu} = -4\pi v^\mu(\rho v^\nu)_{:\nu} - 4\pi\rho v^\nu v^\mu{}_{:\nu}$$

$4\pi E^{\mu\nu} = -F^\mu{}_\rho F^{\nu\rho} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$; $F_{\rho\sigma:\nu} + F_{\sigma\nu:\rho} + F_{\nu\rho:\sigma} = 0$.

$$\begin{aligned} \Rightarrow 4\pi E^{\mu\nu}{}_{:\nu} &= -F^\mu{}_{\rho:\nu}F^{\nu\rho} - F^\mu{}_\rho F^{\nu\rho}{}_{:\nu} + \frac{1}{2}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma:\nu} \\ &= -F^{\mu\rho}F^\nu{}_{\rho:\nu} - g^{\mu\sigma}F^{\nu\rho}F_{\sigma\rho:\nu} + \frac{1}{2}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma:\nu} \\ &= -F^{\mu\rho}F^\nu{}_{\rho:\nu} - \frac{1}{2}g^{\mu\sigma}(F^{\nu\rho}F_{\sigma\rho:\nu} - F^{\nu\rho}F_{\sigma\nu:\rho}) - \frac{1}{2}\overset{\nu\leftrightarrow\sigma}{g^{\mu\sigma}}F^{\rho\nu}F_{\rho\nu:\sigma} \\ &= -F^{\mu\rho}F^\nu{}_{\rho:\nu} - \frac{1}{2}g^{\mu\sigma}F^{\nu\rho}(\underbrace{-F_{\rho\sigma:\nu} - F_{\sigma\nu:\rho} - F_{\rho\nu:\sigma}}_{=0}) \\ &= F^\mu{}_\rho F^{\rho\nu}{}_{:\nu} \underset{②:\ F^{\rho\nu}{}_{:\nu} = 4\pi J^\rho}{=} 4\pi F^{\mu\alpha}J_\alpha \end{aligned}$$

$$\Rightarrow v^\mu(\rho v^\nu)_{:\nu} + \rho v^\nu v^\mu{}_{:\nu} + F^{\mu\alpha}J_\alpha = 0.$$

$\times v_\mu$ $\hookrightarrow$ $(\rho v^\nu)_{:\nu} + \rho v^\nu \underbrace{v_\mu v^\mu{}_{:\nu}}_{} + F^{\mu\alpha}v_\mu J_\alpha = 0$.

geodesic: $v_\nu v^\nu{}_{:\sigma} = 0$.

$$\hookrightarrow (\rho v^\nu)_{:\nu} = -F^{\mu\alpha}v_\mu J_\alpha \underset{J_\alpha = \sigma v_\alpha}{=} -F^{\mu\alpha}\sigma v_\mu v_\alpha = -F^{[\mu\alpha]}\sigma v_{(\mu}v_{\alpha)} = 0.$$

$$\Rightarrow \rho v^\nu v^\mu{}_{:\nu} + F^{\mu\alpha}J_\alpha = 0 \Leftrightarrow \rho v^\nu v_{\mu:\nu} + F_{\mu\nu}J^\nu = 0 \Leftrightarrow ③. \quad \square.$$

▷ Why not indep.? ~ The comprehensive action principle

▷ $\delta(I_g + I') = 0$, I_g ~ the gravitational action. I' ~ other fields.

- $I_g \equiv \int \mathcal{L}\, d^4x = \int \frac{1}{16\pi} L \sqrt{}\, d^4x$, $L \equiv g^{\mu\nu}(\Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho})$ — 需要曲率R分部积分，舍去表面项后的量. $L = L(g_{\alpha\beta}, g_{\alpha\beta,\nu})$.

$$\Rightarrow \delta I_g = \int \left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\delta g_{\alpha\beta,\nu}\right) d^4x \simeq \int \left[\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu}\right]\delta g_{\alpha\beta}\, d^4x = -\frac{1}{16\pi}\int d^4x\, (R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)\sqrt{}\,\delta g_{\alpha\beta}$$

- Let ϕ_n denote the other field quantities (比如 电磁势 K_μ)

  $I' = \int \mathcal{L}' d^4x$, $\mathcal{L}' = \mathcal{L}'(\phi_n, \phi_{n,\rho}, \dots)$

形式上 $\Rightarrow \delta(I_g + I') = \int (p^{\mu\nu}\delta g_{\mu\nu} + \sum_n X^n \delta\phi_n)\sqrt{}\, d^4x$, with $p^{\mu\nu} = p^{\nu\mu}$.

$\overset{\delta(I_g + I') = 0}{\Longrightarrow}$ $p^{\mu\nu} = 0$, $X^n = 0$ $\Rightarrow$ $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - 16\pi N^{\mu\nu} = 0$, just Einstein eqn with $T^{\mu\nu} = -2N^{\mu\nu}$.

$p^{\mu\nu}$: from $\mathcal{L}$, from $\mathcal{L}'$, say $N^{\mu\nu} = \frac{\partial\mathcal{L}'}{\partial g_{\mu\nu}}$

For consistency, $N^{\mu\nu}{}_{:\nu} = 0$.