经典力学 概述

参考：(1) 高显 经典力学讲义
(2) 朗道《力学》
(3) 朗道《力学》解读 韩国兴

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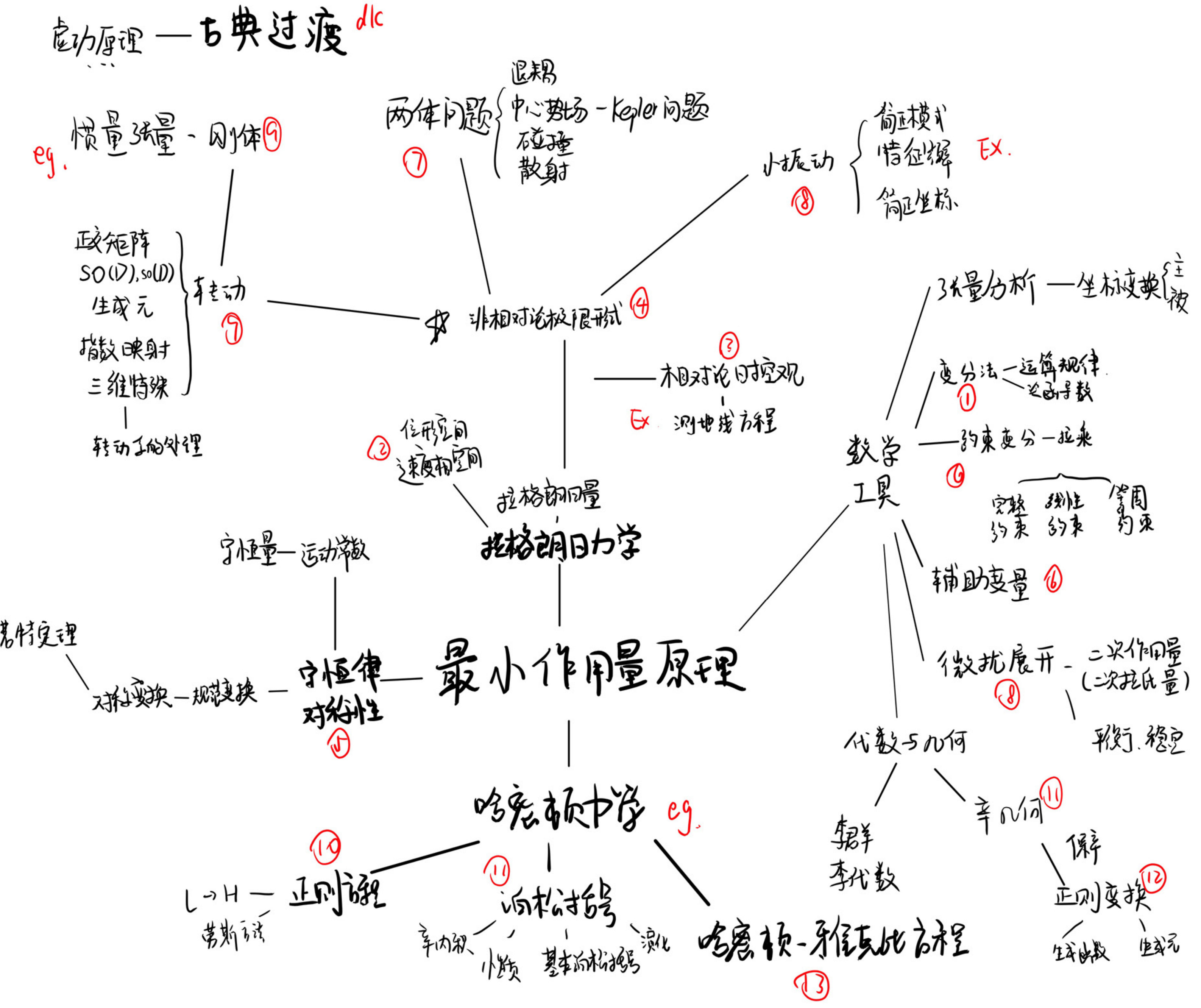
2023/12/12



(无约束) 变分法要点：

1、变分：$\delta f(t) := \tilde{f}(t) - f(t)$

2. 运算规则：同微分；$\delta(df) = d(\delta f)$ ；$\delta S = \delta \int dt\, L = \int dt\, \delta L$.

3. 泛函导数：

$$\begin{aligned} \mathrm{d}F &= \sum_n \; \frac{\partial F}{\partial x_n} \; \mathrm{d}x_n \\ \delta S &= \int \mathrm{d}t \; \frac{\delta S}{\delta f(t)} \; \delta f(t) \end{aligned}$$

图 1.6: 泛函导数与多元函数偏导数的类比。

4. 变分的计算流程.

对于 $S[f] = \int dt\, L(t, f, f')$,

$\delta S = \int dt\, \delta L(t, f, f')$

$= \int dt \left(\frac{\partial L}{\partial f} \delta f + \frac{\partial L}{\partial f'} \delta f' \right)$ ↘ 分部积分，去边界项.

$\simeq \int dt \left(\frac{\partial L}{\partial f} \delta f - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right) \delta f \right)$

对多变量、多元的推广是直接的.

5、泛函极值条件.

$\delta S = 0 \Leftrightarrow \frac{\delta S}{\delta f} = 0$ 为泛函极值的必要条件. 将导出E-L方程.

位形流形要点：

1、概念：位形空间（流形）；世界线～点粒子；广义坐标；广义速度；$\left\{ \begin{array}{l} \text{相空间} \\ \text{速度相空间} \end{array} \right.$. 自由度 = # 独立广义坐标变分

2. 广义坐标的变换（点变换）：$\{q^a\} \to \{\tilde{q}^a\}$, $\tilde{q}^a = \tilde{q}^a(t, \vec{q})$. 可逆条件: $\det\left(\frac{\partial \tilde{q}^a}{\partial q^a}\right) \neq 0$.

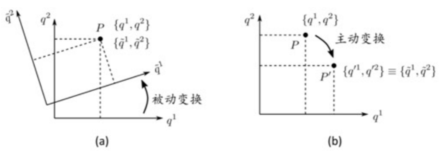


图 2.7: 坐标变换的两种等价观点：(a) 被动观点，(b) 主动观点。

可诱导广义速度的变换 $\dot{\tilde{q}}^a = \frac{d}{dt}\tilde{q}^a = \frac{\partial \tilde{q}^a}{\partial q^b} \dot{q}^b$

$\Rightarrow \frac{\partial \dot{\tilde{q}}^a}{\partial \dot{q}^b} = \frac{\partial \tilde{q}^a}{\partial q^b}$.

该变换下运动方程形式不变.

3. 多数物理系统可由 $q^a(t)$ 与 $\dot{q}^a(t)$ 唯一确定. ～ E-L方程为二阶DE, $\ddot{q}^a(t) = F(q(t), \dot{q}(t))$.

4. 约束.

$\left\{ \begin{array}{l} \text{定常} \\ \text{非定常} \end{array} \right.$ $\left\{ \begin{array}{l} \text{完整} \\ \text{非完整} \end{array} \right.$ - 不可积微分约束
- 不等式
...

完整约束：$\phi(t, \vec{q}) = 0$. $\delta\phi = \frac{\partial \phi}{\partial q^a} \delta q^a$, i.e. $\nabla\phi$ 与 $\delta\vec{q}$ 正交.

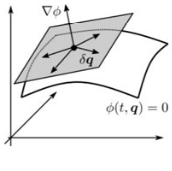


图 2.13: 完整约束作为位形空间中的曲面。

达朗贝尔原理要点：

$\sum_\alpha (\vec{F}_{(\alpha)} - m_{(\alpha)} \ddot{\vec{x}}_{(\alpha)}) \cdot \delta \vec{x}_{(\alpha)} = 0$ ，假设 $\sum_\alpha \vec{N}_{(\alpha)} \cdot \delta \vec{x}_{(\alpha)} = 0$

↑主动力 ↑惯性力

理想约束. 解释：$\delta\vec{x}$ ∥ 约束面，$\vec{N}$ 沿约束面法向.

平衡 ↳ $\sum_\alpha \vec{F}_{(\alpha)} \cdot \delta \vec{x}_{(\alpha)} = 0$. 虚功原理.

相对论时空观要点:

1、

空间延展维度	0	1	2	3
空间中的对象	粒子	弦	膜	场
时空中的对象	世界线	世界面	世界体	场
理论	经典力学	弦理论	膜理论	场论

2. 度规 $g_{\mu\nu}$. 空间属性的表征. (无挠, 黎曼流形中, $g_{\mu\nu}$ 对称, 非退化).

线元: $ds^2 = g_{\mu\nu}dq^\mu dq^\nu = d\vec{q}^T g\, d\vec{q}$, 无穷小矢距离的积.

逆度规 $g^{\mu\nu} \equiv (g^{-1})_{\mu\nu}$. $g^{\mu\nu}g_{\nu\sigma} = g^\mu{}_\sigma = \delta^\mu{}_\sigma$.

3. 张量: 升降指标; 协变 & 逆变; 缩并; 以度规定义内积.

4. 参考系. 观测者. 惯性系. 相对性原理.

固有时 τ: 世界线 $x^\mu = x^\mu(t)$, t为参数. 取参数为τ, 可使得 $|ds| = cd\tau$.

最小作用量原理要点: $\delta S = 0$.

· 作用量是标量.

量纲: [作用量] = [时间] · [能量]
= [空间] · [动量]
= [角度] · [角动量]
= [普朗克常数$\hbar$].

广义坐标变换: 作用量积分参数的重参数化.

对于点粒子, 即 $t \to \tilde{t} = \tilde{t}(t)$. 对于场, 则是 $x^\mu \to \tilde{x}^\mu(x^\rho)$.

标量 ~ 广义坐标变换下不变.

· 常见作用量.

① 自由粒子: (4D) $S = -mc\int |ds| = \underbrace{\overset{\text{凑量纲的系数}}{\overbrace{-mc}}\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}}_{\text{世界线长}}$.

(ps: 号差为+1. i.e. $\eta_{\mu\nu} = \begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix}$).

参数化↓

$$S = -mc\int dt\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \xrightarrow[ds^2 = -c^2d\tau^2]{\text{取参数为}\tau} S = -mc\int d\tau\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \quad . \quad \frac{dx^\mu}{d\tau} \equiv u^\mu \text{ 为4-速度.}$$

疑难1

此时对 x^μ 变分, 取何参数进行参数化未作考虑, 只关注 δx^μ 引起的 δS.

i.e. $\delta S = F(\delta x^\mu)$

导出其运动方程:

$$\delta S = -mc\int d\tau\,\delta\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \overset{\text{对}x^\mu\text{变分}}{=} -mc\int d\tau\frac{1}{\underline{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}}\frac{d}{d\tau}(\delta x^\mu)\frac{dx^\nu}{d\tau}$$

由于参数τ的选取, $\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -c$.

$$= m\int d\tau\frac{dx^\nu}{d\tau}\delta\left(\frac{dx^\mu}{d\tau}\right) \simeq -m\int dt\frac{d^2x^\nu}{d\tau^2}\delta x^\mu.$$

时空分离: (3D) $S = -mc\int\sqrt{c^2dt^2 - \delta_{ij}dx^idx^j} = -mc^2\int dt\sqrt{1 - \frac{1}{c^2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}$, $\frac{dx^i}{dt} \equiv v^i$, 3-速度.

$\Rightarrow S = \int dt\,L$, $L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$. $\longrightarrow$ $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma$.

3-动量: $P_i \equiv \frac{\partial L}{\partial v^i} = \gamma m v_i = m u_i$, u_i为u_μ的空间分量.

非相对论极限: $S = -mc^2\int dt\left(1 - \frac{1}{2}\frac{v^2}{c^2} + \dots\right) = \underset{\overset{\uparrow}{\text{const}}}{-mc^2\int dt} + \int dt\frac{1}{2}mv^2 + \dots \simeq \int dt\,T$.

② 指数标量场 $S = -mc\int e^\Phi|ds| = -mc^2\int dt\,e^\Phi\sqrt{1-\frac{v^2}{c^2}}$.

$$\xrightarrow[\text{弱场: }\Phi \equiv \frac{V}{mc^2} \ll 1]{\text{低速. }\frac{v^2}{c^2} \ll 1} S = -mc^2\int dt\left(1 + \frac{V}{mc^2} + \dots\right)\left(1 - \frac{1}{2}\frac{v^2}{c^2} + \dots\right)$$

$\simeq \int dt(T - V)$

T-V 形式. 负号源于 $\eta_{\mu\nu}$.

③ 带电粒子: $I_{ele}=\int A_\mu dx^\mu$.

电磁场: $I_{em}=\int F_{\mu\nu}F^{\mu\nu}d^4x$. (省去系数).

④ 引力场: $S=-mc\int|ds|=-mc\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$

· 非相对论极限下的拉氏量形式: $L=T-V=\sum_\alpha\frac{1}{2}m_{(\alpha)}\dot{\vec{x}}_{(\alpha)}^2-V(\vec{x}_{(\alpha)})=\frac{1}{2}G_{ab}\dot{q}^a\dot{q}^b+X_a\dot{q}^a+Y$

(N粒子系统) 其中 $\vec{x}_{(\alpha)}=\vec{x}_{(\alpha)}(t,\vec{q})$, 故 $\dot{\vec{x}}_{(\alpha)}=\frac{\partial\vec{x}_{(\alpha)}}{\partial q^a}\dot{q}^a+\frac{\partial\vec{x}_{(\alpha)}}{\partial t}$. $G_{ab}=\sum_\alpha m_{(\alpha)}\frac{\partial\vec{x}_{(\alpha)}}{\partial q^a}\cdot\frac{\partial\vec{x}_{(\alpha)}}{\partial q^b}$, $X_a=\sum_\alpha\frac{\partial\vec{x}_{(\alpha)}}{\partial q^a}\cdot\frac{\partial\vec{x}_{(\alpha)}}{\partial t}$, $Y=\sum_\alpha(\frac{\partial\vec{x}_{(\alpha)}}{\partial t})^2$

Ex1. 推导弯曲空间中自由粒子的运动方程.

$$S=-mc\int|ds|=-mc\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}=-mc\int d\tau\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

$$\delta S=-mc\int d\tau\,\delta\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

$$=-mc\int d\tau\frac{1}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}\delta(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau})$$

$$=-m\int d\tau\,[\,g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}(\delta x^\nu)+\frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta g_{\mu\nu})$$

$$\simeq-m\int d\tau\,[\,-\frac{d}{d\tau}(g_{\mu\nu}\frac{dx^\mu}{d\tau})\delta x^\nu+\frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}g_{\mu\nu,\sigma}\delta x^\sigma\,]$$

$$=m\int d\tau\,[(g_{\mu\nu,\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}+g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2})\delta x^\nu-\frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}g_{\mu\sigma,\nu}\delta x^\nu\,]$$

$\delta S=0$

$$\Rightarrow g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}-\frac{1}{2}g_{\mu\sigma,\nu}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}+g_{\mu\nu,\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}=0.$$

记 $\frac{dx^\mu}{d\tau}\equiv u^\mu$, 且 $g_{\mu\nu,\sigma}u^\sigma u^\mu=\frac{1}{2}(g_{\mu\nu,\sigma}u^\sigma u^\mu+g_{\sigma\nu,\mu}u^\sigma u^\mu)$

故 $g_{\mu\nu}\frac{du^\mu}{d\tau}+\frac{1}{2}(g_{\mu\nu,\sigma}+g_{\sigma\nu,\mu}-g_{\mu\sigma,\nu})u^\sigma u^\mu$

同乘 $g^{\nu\lambda}$ ↳ $\frac{du^\lambda}{d\tau}+\frac{1}{2}g^{\nu\lambda}(g_{\mu\nu,\sigma}+g_{\sigma\nu,\mu}-g_{\mu\sigma,\nu})u^\sigma u^\mu\equiv\frac{du^\lambda}{d\tau}+\Gamma^\lambda_{\mu\sigma}u^\mu u^\sigma=0.$

//

对称性与守恒律要点:

↗ 注意只对真实运动成立. 故不可代回原始的拉氏量.

1. 运动常数 记真实运动为 $\{\vec{q}_u(t)\}$, 则 $C(t,q^a,\dot{q}^a)$ s.t. $\frac{dC(t,\vec{q}(t),\dot{\vec{q}}(t))}{dt}\Big|_{\vec{q}_u}=0$ 为运动常数.

不显含 t ~ 整体运动常数. S自由度系统 有 2S-1 个 独立的 整体运动常数.

可加性运动常数 ~ 守恒量 ~ 时空对称性.

- 某个 q^a, s.t. $\frac{\partial L}{\partial q^a}=0$ (循环坐标) $\longrightarrow$ $\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}^a})=0$, i.e. $p_a\equiv\frac{\partial L}{\partial\dot{q}^a}=$ const ~ 广义动量守恒.
- 若 $\frac{\partial L}{\partial t}=0$, 由 EL方程等价形式 $\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}^a}\dot{q}^a-L)+\frac{\partial L}{\partial t}=0$ 得 $h(t,\vec{q},\dot{\vec{q}})\equiv\frac{\partial L}{\partial\dot{q}^a}\dot{q}^a-L(t,\vec{q},\dot{\vec{q}})=$ const. ~ 能量函数守恒 ~ 保守系.

对非定常系统, $h\neq E=T+V$.

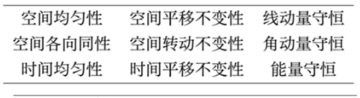
2. 规范变换: $\tilde{L}=L+\frac{dF}{dt}\Leftrightarrow\tilde{L}\simeq L$, $F=F(t,\vec{q})$

$\tilde{S}=S+F|_{t_1}^{t_2}\Leftrightarrow\tilde{S}\simeq S$, $\delta\tilde{S}=\delta S$.

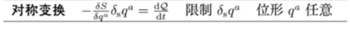
$\tilde{p}_a=\frac{\partial\tilde{L}}{\partial\dot{q}^a}=p_a+\frac{\partial F}{\partial q^a}$, 有一定任意性

$\tilde{h}=\frac{\partial\tilde{L}}{\partial\dot{q}^a}\dot{q}^a-\tilde{L}=h-\frac{\partial F}{\partial t}$.

3.



对称性	不变性	守恒律
空间均匀性	空间平移不变性	线动量守恒
空间各向同性	空间转动不变性	角动量守恒
时间均匀性	时间平移不变性	能量守恒



运动方程	$\frac{\delta S}{\delta q^a}=0$	限制 q^a	变分 δq^a 任意
对称变换	$-\frac{\delta S}{\delta q^a}\delta_s q^a=\frac{\mathrm{d}Q}{\mathrm{d}t}$	限制 $\delta_s q^a$	位形 q^a 任意

约束变分要点：（构造作用量 s.t. $\delta S=0$ 导出正确的运动方程）.

1. 完整约束 $\phi(t,\vec{q})=0$： 泛函极值条件：$\frac{\delta S}{\delta q^a}=-\lambda\frac{\partial\phi}{\partial q^a}$. （由于 $\delta\phi=\frac{\partial\phi}{\partial q^a}\delta q^a=0$，故 δq^a 不独立）

构造 $\tilde{S}[\vec{q},\lambda]=\int dt\,\tilde{L}=\int dt\,[L(t,\vec{q},\dot{\vec{q}})+\lambda(t)\,\phi(t,\vec{q})]$,

则 $\delta\tilde{S}=0$ 给出 $\frac{\partial L}{\partial q^a}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^a}=-\lambda\frac{\partial\phi}{\partial q^a}$; $\phi=0$. 正是泛函极值条件+约束方程.

2. 一般的非完整系统不能纳入最小作用量原理框架.

特例：$\phi(t,\vec{q},\dot{\vec{q}})=A_a(t,\vec{q})\dot{q}^a+B(t,\vec{q})=0$.

$\Leftrightarrow \underline{\phi\,dt=A_a\,dq^a+B\,dt=0}\xrightarrow[\text{任一瞬时}]{dt=0}\underline{A_a\delta q^a=0}$. 泛函极值条件：$\frac{\delta S}{\delta q^a}=-\lambda A_a$

故 $\delta\tilde{S}[\vec{q}]=\int dt\,\delta\tilde{L}=\int dt\,[\delta L(t,\vec{q},\dot{\vec{q}})+\lambda A_a\delta q^a]$

3. 等周约束：$\int dt\,\phi(t,\vec{q},\dot{\vec{q}})=C$

$\tilde{S}[\vec{q}]=\int dt\,(L+\lambda\phi)=\int dt\,(L(t,\vec{q},\dot{\vec{q}})+\lambda\phi(t,\vec{q},\dot{\vec{q}}))$

疑难2:

* 对于 $\phi(t,\vec{q},\dot{\vec{q}})=0$，$\delta\phi=\frac{\partial\phi}{\partial q^a}\delta q^a+\underbrace{\frac{\partial\phi}{\partial\dot{q}^a}\frac{d}{dt}(\delta q^a)}=0$. 不能得到 $\frac{\partial\phi}{\partial q^a}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{q}^a}\right)=0$.

$\downarrow$

因为这一项未必是0. $\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{q}^a}\delta q^a\right)-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{q}^a}\right)\delta q^a$

PS：最小作用量原理是 $\delta S=0$. 并不意味着 $\delta L=0$.

- 辅助变量：只有广义坐标本身出现在拉氏量中. 拉格朗日乘子为其中一个特例.
  本身无独立的动力学方程，演化由动力学变量完全决定.
  若 $L=L(t,q,\dot{q};\chi)$，则 χ 的运动方程 $\frac{\partial L}{\partial\chi}=0$ 为 χ 的代数方程，可解得 $\chi=\chi(t,q,\dot{q})$.
  可回代入 q 的运动方程得：$\left(\frac{\partial L(t,q,\dot{q};\chi)}{\partial q}-\frac{d}{dt}\frac{\partial L(t,q,\dot{q},\chi)}{\partial q}\right)\Big|_{\chi=\chi(t,q,\dot{q})}=0$.

○ 作用量的几种等价形式:

$$S[q]=\int dt\,L(t,q,\dot{q})$$

对λ变分回代

$$S[q,\lambda,\chi]=\int dt\,[L(t,q,\chi)+\lambda(\chi-\dot{q})]$$

对χ变分，代换λ

对χ变分回代

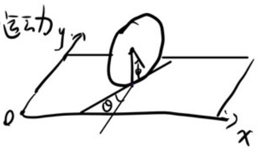
$$S[q,\chi]=\int dt\left[L(t,q,\chi)-\frac{\partial L(t,q,\chi)}{\partial\chi}(\chi-\dot{q})\right]$$

降阶：$S[q]=\int dt\,L(t,q,\dot{q},\ddot{q})\Leftrightarrow S[q,\chi,\lambda]=\int dt\,[L(t,q,\dot{q},\dot{\chi})+\lambda(\chi-\dot{q})]$

对λ变分回代.

EX2. 自由圆盘的运动

（纯滚动）.



选 $\{x,y,\theta,\phi\}$.

约束：由纯滚条件，$\begin{cases}R\dot{\phi}\sin\theta-\dot{x}=0.\\ R\dot{\phi}\cos\theta-\dot{y}=0.\end{cases}\Rightarrow\begin{cases}R\ddot{\phi}\sin\theta+R\dot{\phi}\dot{\theta}\cos\theta-\ddot{x}=0\\ R\ddot{\phi}\cos\theta-R\dot{\phi}\dot{\theta}\sin\theta-\ddot{y}=0\end{cases}$

$L=T-V=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{4}mR^2\dot{\phi}+\frac{1}{8}mR^2\dot{\theta}^2$

$\delta\tilde{S}[x,y,\theta,\phi,\alpha,\beta]=\int dt\,[\delta L+\alpha(\delta x-\sin\theta R\delta\phi)+\beta(\delta y-\cos\theta R\delta\phi)]=0$

$$\Rightarrow\begin{cases}m\ddot{x}=\alpha\\ m\ddot{y}=\beta\\ \ddot{\theta}=0\ \longrightarrow\ \text{方向不变}\\ \frac{1}{2}mR\ddot{\phi}+\alpha\sin\theta+\beta\cos\theta=0.\end{cases}$$

$\downarrow$

$\frac{1}{2}R\ddot{\phi}+\ddot{x}\sin\theta+\ddot{y}\cos\theta=0.\Rightarrow\frac{1}{2}R\ddot{\phi}+R\ddot{\phi}=0\Rightarrow\ddot{\phi}=0.$

$$I=\int d^3x\,\rho(\vec{x})\begin{pmatrix}y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz\\ -xz & -yz & x^2+y^2\end{pmatrix}$$

$$=\int_0^{2\pi}d\theta\int_0^R r\,dr\int dz\,\frac{m}{\pi R^2}\delta(z)\begin{pmatrix}r^2\sin^2\theta+z^2 & -r^2\sin\theta\cos\theta & -r\cos\theta z\\ -r^2\cos\theta\sin\theta & r^2\cos^2\theta+z^2 & -r\sin\theta z\\ -r\cos\theta z & -r\sin\theta z & r^2\end{pmatrix}$$

$$=\frac{m}{\pi R^2}\int_0^R r^3dr\int_0^{2\pi}d\theta\begin{pmatrix}\sin^2\theta & -\sin\theta\cos\theta & 0\\ -\sin\theta\cos\theta & \cos^2\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\frac{m}{\pi R^2}\cdot\frac{R^4}{4}\begin{pmatrix}\pi & 0 & 0\\ 0 & \pi & 0\\ 0 & 0 & 2\pi\end{pmatrix}$$

$$=\frac{mR^2}{4}\begin{pmatrix}1 & & \\ & 1 & \\ & & 2\end{pmatrix}.$$

两体与中心势场问题要点:

1. 两体问题的退耦: $T=\frac{1}{2}(\dot{\vec{x}}_c\ \dot{\vec{r}})\begin{pmatrix}m_t & \\ & m_r\end{pmatrix}\begin{pmatrix}\dot{\vec{x}}_c\\ \dot{\vec{r}}\end{pmatrix}$, $m_r=\frac{m_1m_2}{m_1+m_2}$ 为约化质量

质心系中的运动可看作中心势场问题. $L_r=\frac{1}{2}m_r\dot{\vec{r}}^2-V(r)$.

2. 中心势场问题.

运动方程: $m\ddot{\vec{r}}=-\nabla V=\frac{dV}{dr}\hat{r}$.

由中心对称性, 取$\{r,\phi\}$, $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)-V(r)$

ϕ为循环坐标, $P_\phi=\frac{\partial L}{\partial\dot{\phi}}=mr^2\dot{\phi}=J=const.$ $\Rightarrow \dot{\phi}=\frac{d\phi}{dt}=\frac{J}{mr^2}$.

又 $\frac{\partial L}{\partial t}=0$, 故 $h=\frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)+V(r)\equiv E=const.$

$\Rightarrow E=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}+V(r)\equiv\frac{1}{2}m\dot{r}^2+V_{eff}(r)$ $V_{eff}(r)=E$ 给出边界点.

可得 $\frac{dr}{dt}=\pm\sqrt{\frac{2}{m}(E-V(r))-\frac{J^2}{m^2r^2}}$.

$\Rightarrow t=\pm\int\frac{dr}{\sqrt{\frac{2}{m}(E-V(r))-\frac{J^2}{m^2r^2}}}$ 即 $t=t(r)$. 可反解出 $r=r(t)$

又 $\frac{d\phi}{dr}=\frac{d\phi/dt}{dr/dt}$. 有 $\frac{mr^2}{J}d\phi=\pm\frac{dr}{\sqrt{\frac{2}{m}(E-V(r))-\frac{J^2}{m^2r^2}}}$. 可解出 $\phi=\phi(r)$ / $r=r(\phi)$.

对有界运动, $r_1\to r_2\to r_1$ 过程中,

$$\Delta\phi=2\int_{r_1}^{r_2}\frac{J}{r^2}\frac{dr}{\sqrt{2m(E-V(r))-\frac{J^2}{r^2}}}.$$ 当 $\Delta\phi=2\pi\frac{m}{n}$ 时轨道闭合.

3. 开普勒问题: $V(r)=-\frac{\alpha}{r}$, $\alpha>0$. $V_{eff}=-\frac{\alpha}{r}+\frac{J^2}{2mr^2}$.

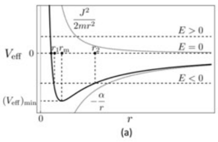


图 8.6: (a) 开普勒问题中径向运动的有效势能。

$$\phi(r)=\int\frac{J}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}(E+\frac{\alpha}{r})-\frac{J^2}{m^2r^2}}}$$

记 $P:=\frac{J^2}{m\alpha}$, $e=\sqrt{1+\frac{2EJ^2}{m\alpha^2}}$.

得 $\phi=\int dr\frac{J^2}{m\alpha r^2}\frac{1}{\sqrt{\frac{2}{m}\frac{J^2}{\alpha^2}(E+\frac{\alpha}{r})-\frac{J^4}{\alpha^2m^2r^2}}}=\int dr\frac{P}{r^2}\frac{1}{\sqrt{e^2-(1-\frac{P}{r})^2}}\overset{u=1-\frac{P}{r}}{=\!=\!=}\int\frac{du}{\sqrt{e^2-u^2}}=\arcsin(\frac{u}{e})+\phi_0$

取 $\phi_0=\frac{\pi}{2}$, 则 $u=-e\cos\phi$, i.e. $r=\frac{P}{1+e\cos\phi}$ 为圆锥曲线.

当 $E<0$ 时, $e<1$, $V_{eff}=E$ 可解得 $r_1=\frac{P}{1-e}$, $r_2=\frac{P}{1+e}$. 半长轴 $a=\frac{r_1+r_2}{2}=\frac{P}{2(1-e^2)}=\frac{\alpha}{-2E}=\frac{\alpha}{2|E|}$, $b=\frac{P}{\sqrt{1-e^2}}=\frac{J}{\sqrt{2m|E|}}$.

周期 $T=\frac{\pi ab}{\frac{1}{2}\dot{\phi}r^2}=2\pi a^{\frac{3}{2}}\sqrt{m/\alpha}$ ~ kepler第三定律.

↙
掠面速度.

- LRL矢量: $\vec{A}\equiv\vec{p}\times\vec{J}-m\alpha\frac{\vec{r}}{r}$.
  利用 $\vec{r}\cdot\vec{A}$ 也可求轨道.

$$\frac{d\vec{A}}{dt}=\frac{d\vec{p}}{dt}\times(\vec{r}\times\vec{p})+m\alpha\frac{\vec{r}}{r^2}\dot{r}-m\alpha\frac{\dot{\vec{r}}}{r}$$
$$=-\frac{\alpha}{r^3}\vec{r}\times(\vec{r}\times\vec{p})+m\alpha\frac{\vec{r}}{r^2}\dot{r}-m\alpha\frac{\dot{\vec{r}}}{r}$$
$$=-\frac{\alpha}{r^3}[(\vec{r}\cdot\vec{p})\vec{r}-r^2\vec{p}]+m\alpha\frac{\vec{r}}{r^2}\dot{r}-m\alpha\frac{\dot{\vec{r}}}{r}=0.$$

- $m\frac{d\vec{p}}{dt}=-\nabla U=-\frac{dV}{dr}\frac{\vec{r}}{r}=-\frac{\alpha\vec{r}}{r^3}$

其中 $\vec{r}\cdot\vec{p}=m\vec{r}\cdot\dot{\vec{r}}=m\,g_{ij}r^i\dot{r}^j$

而 $r\dot{r}=\sqrt{g_{ij}r^ir^j}(\sqrt{g_{\mu\nu}r^\mu r^\nu})'$

$=r\frac{1}{r}g_{\mu\nu}r^\mu\dot{r}^\nu=g_{\mu\nu}r^\mu\dot{r}^\nu=\vec{r}\cdot\dot{\vec{r}}$

$\Rightarrow\vec{r}\cdot\vec{p}=m\vec{r}\cdot\dot{\vec{r}}=mr\dot{r}$.

4. $U=-\frac{\alpha}{r}+\delta U$ 时, δU 引起的进动.

$r_1\to r_2\to r_1$ 过程中, $\Delta\phi=2\int_{r_1}^{r_2}\frac{J}{r^2}\frac{dr}{\sqrt{2m(E-V(r))-\frac{J^2}{r^2}}}=-2\frac{\partial}{\partial J}\int_{r_1}^{r_2}\sqrt{2m(E-V(r))-\frac{J^2}{r^2}}\,dr$.

代入 $U=-\frac{\alpha}{r}+\delta U$, δU 为小量, 故 $\sqrt{2m(E-U)-\frac{J^2}{r^2}}=\sqrt{2m(E+\frac{\alpha}{r}-\delta U)-\frac{J^2}{r^2}}\approx\sqrt{2m(E+\frac{\alpha}{r})-\frac{J^2}{r}}-\frac{m\delta U}{\sqrt{2m(E+\frac{\alpha}{r})-\frac{J^2}{r^2}}}$ (保留到1阶).

$\Rightarrow\Delta\phi=\underset{\uparrow}{2\pi}+\delta\phi$, $\delta\phi=\frac{\partial}{\partial J}\int_{r_1}^{r_2}\frac{2m\delta U\,dr}{\sqrt{2m(E+\frac{\alpha}{r})-\frac{J^2}{r}}}$ 为进动角.

第一项就是kepler运动.

$V=-\frac{\alpha}{r}$ 时有 $\frac{mr^2}{J}d\phi=\pm\frac{dr}{\sqrt{\frac{2}{m}(E-V(r))-\frac{J^2}{m^2r^2}}}$, 故近似有 $\frac{2m\delta U\,dr}{\sqrt{2m(E+\frac{\alpha}{r})-\frac{J^2}{r^2}}}=2m\delta U\frac{r^2}{J}d\phi$. $\delta\phi=\frac{\partial}{\partial J}\left(\frac{2m}{J}\int_0^\pi r^2\delta U\,d\phi\right)$

广义相对论第一项修正: $\delta U=\frac{\beta}{r^3}$, i.e. $U=-\frac{\alpha}{r}+\frac{\beta}{r^3}$.

此时 $\int_0^\pi r^2\delta U\,d\phi=\beta\int_0^\pi\frac{1}{r}d\phi=\frac{\beta}{P}\int_0^\pi(1+e\cos\phi)d\phi=\frac{\beta\pi}{P}=\frac{\beta\pi m\alpha}{J^2}$.

$\delta\phi=\frac{\partial}{\partial J}\left(\frac{2m}{J}\frac{\beta\pi m\alpha}{J^2}\right)=2m^2\beta\pi\alpha(-\frac{3}{J^4})=\frac{-6\pi\beta}{P^2\alpha}$ 为进动角.

微扰展开要点：

概念：在已知解附近线性化，得到该近似的线性系统的新解.

对多自由度系统，$\{q^a\}$，记扰动为 $Q^a(t)$，一组特解为 $\{\bar{q}^a(t)\}$，则 $q^a(t)=\bar{q}^a(t)+\epsilon Q^a(t)$.

作用量 $S[\vec{q}]=S[\vec{\bar{q}}+\epsilon\vec{Q}]=\int dt\, L(t,\vec{\bar{q}}+\epsilon\vec{Q},\dot{\vec{\bar{q}}}+\epsilon\dot{\vec{Q}})$

$$=\int dt\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left[\left(Q^a\frac{\partial}{\partial q}+\dot{Q}^a\frac{\partial}{\partial\dot{q}}\right)^n L\right]\Big|_{\vec{\bar{q}}}$$

$$=\underset{\substack{\uparrow\\ \text{背景作用量}}}{S_0[\vec{\bar{q}}]}+\epsilon\underset{\substack{|||\\ 0}}{S_1[\vec{Q}]}+\epsilon^2\underset{\substack{\downarrow\\ \text{自由部分,}\\ \text{扰动的领头阶近似.}}}{\underline{S_2[\vec{Q}]}}+\underbrace{\epsilon^3 S_3[\vec{Q}]+\dots}_{\text{相互作用.}}$$

☆ 二次作用量：$S_2[\vec{Q}]=\int dt\left(\frac{1}{2}G_{ab}\dot{Q}^a\dot{Q}^b+N_{ab}\dot{Q}^aQ^b+\frac{1}{2}M_{ab}Q^aQ^b\right)=\int dt\left(\frac{1}{2}G_{ab}\dot{Q}^a\dot{Q}^b+\frac{1}{2}F_{ab}\dot{Q}^aQ^b-\frac{1}{2}W_{ab}Q^aQ^b\right)$

此处 $\underset{(\ )}{G_{ab}}=\frac{\partial^2L}{\partial\dot{q}^a\partial\dot{q}^b}\Big|_{\vec{\bar{q}}}$，$N_{ab}=\frac{\partial^2L}{\partial\dot{q}^a\partial q^b}\Big|_{\vec{\bar{q}}}$，$\underset{(\ )}{M_{ab}}=\frac{\partial^2L}{\partial q^a\partial q^b}\Big|_{\vec{\bar{q}}}$，$\underset{[\ ]}{F_{ab}}=N_{ab}-N_{ba}$，$W_{ab}=-M_{ab}+\dot{N}_{ba}$

- 平衡位形：$\bar{q}^a=const$.

由拉氏量形式 $L=\frac{1}{2}\tilde{G}_{ab}(\vec{q})\dot{q}^a\dot{q}^b-V(\vec{q})$ 可知 $N_{ab}\propto\dot{q}^b$，故当 $\bar{q}^a=const$ 时有 $N_{ab}=0$，

定常系统中，$L_2=\frac{1}{2}\underset{\substack{\uparrow\\ \text{正定}\\ \text{(动能要求)}}}{G_{ab}}\dot{Q}^a\dot{Q}^b-\frac{1}{2}\underset{\substack{\uparrow\\ \text{若稳定，则 }W_{ab}\text{ 正定.}\\ \text{(势能的黑塞矩阵)}}}{W_{ab}}Q^aQ^b$. $G_{ab}=\frac{\partial^2T}{\partial\dot{q}^a\partial\dot{q}^b}\Big|_{\vec{\bar{q}}}$，$W_{ab}=\frac{\partial^2V}{\partial q^a\partial q^b}\Big|_{\vec{\bar{q}}}$ (注意与前文区分).

应用：小振动问题：（S自由度系统）

$L_2=\frac{1}{2}G_{ab}\dot{q}^a\dot{q}^b-\frac{1}{2}W_{ab}q^aq^b=\frac{1}{2}\dot{\vec{q}}^TG\dot{\vec{q}}-\frac{1}{2}\vec{q}^TW\vec{q}$ （重记扰动为 q^a）.

运动方程为：$G_{ab}\ddot{q}^b+W_{ab}q^b=0 \Leftrightarrow G\ddot{\vec{q}}+W\vec{q}=0$.

试解为 $\vec{q}(t)=\vec{A}e^{-i\omega t}+cc$.

代入得 $(W-\omega^2G)\vec{A}=0$. $\vec{A}$ 为 $(W-\omega^2G)$ 的零本征矢. $\vec{A}$ 非平凡 $\Rightarrow$ $W-\omega^2G$ 退化.

特征方程：$\det(W-\omega^2G)=0$. 其解 ω_α，$\alpha=1,\dots,S$ 为特征频率.

$\Leftrightarrow$ $G^{-1}W\vec{A}=\omega^2\vec{A}$，i.e. $\vec{A}_\alpha$ 为 $G^{-1}W$ 对应于本征值 ω_α^2 的本征矢.

故有解集 $\{\vec{q}_\alpha(t)\}=\{\vec{A}_\alpha e^{-i\omega t}+cc.\}_{\alpha=1}^{S}$，通解 $\vec{q}(t)=\sum_{\alpha=1}^{S}\underset{\substack{\uparrow\\ \text{复常数，由 IC 确定.}}}{\tilde{C}_\alpha}\vec{A}_\alpha e^{-i\omega_\alpha t}+cc.$

- 解耦：使 G，W 对角化

取 $\vec{q}=M\vec{\xi}$. 则 $L_2=\frac{1}{2}(M\dot{\vec{\xi}})^TG(M\dot{\vec{\xi}})-\frac{1}{2}(M\vec{\xi})^TW(M\vec{\xi})$

$$=\frac{1}{2}\dot{\vec{\xi}}^TM^TGM\dot{\vec{\xi}}-\frac{1}{2}\vec{\xi}^TM^TWM\vec{\xi}$$

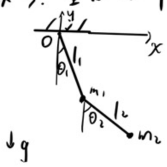
$$\equiv\frac{1}{2}\dot{\vec{\xi}}^T\tilde{G}\dot{\vec{\xi}}-\frac{1}{2}\vec{\xi}^T\tilde{W}\vec{\xi}.$$

以 G 定义 G-内积：$\langle A|B\rangle:=\vec{A}^TG\vec{B}$. 可取 S个 独立 本征矢 s.t. $\langle A_\alpha|A_\beta\rangle=\delta_{\alpha\beta}$.

令 $M=(\vec{A}_1\ \cdots\ \vec{A}_S)$，i.e. $M^a{}_\alpha=A^a{}_\alpha$，$a,\alpha=1,\dots,S$ 即可使 $\tilde{G}$，$\tilde{W}$ 对角化.

简正坐标 $\vec{\xi}=M^{-1}\vec{q}=M^TG\vec{q}$.

EX 3. 重力场中的双摆在平衡位形附近的小振动.



选择：$\{\theta_1,\theta_2\}$.

```
In[116]:= x1 = l1 Sin[θ1[t]]; y1 = l1 Cos[θ1[t]];
                  正弦                  余弦
           x2 = x1 + l2 Sin[θ2[t]]; y2 = y1 + l2 Cos[θ2[t]];
                        正弦                      余弦
```

求拉格朗日量

```
In[118]:= T = 1/2 m1 (D[x1, t]^2 + D[y1, t]^2) + 1/2 m2 (D[x2, t]^2 + D[y2, t]^2) // Simplify
                                                                                    化简

Out[118]= 1/2 (l1^2 (m1 + m2) θ1'[t]^2 + 2 l1 l2 m2 Cos[θ1[t] - θ2[t]] θ1'[t] θ2'[t] + l2^2 m2 θ2'[t]^2)

In[119]:= V = m1 g y1 + m2 g y2

Out[119]= g l1 m1 Cos[θ1[t]] + g m2 (l1 Cos[θ1[t]] + l2 Cos[θ2[t]])

In[122]:= L = T - V // Simplify
                     化简

Out[122]= 1/2 (-2 g (l1 (m1 + m2) Cos[θ1[t]] + l2 m2 Cos[θ2[t]]) +
            l1^2 (m1 + m2) θ1'[t]^2 + 2 l1 l2 m2 Cos[θ1[t] - θ2[t]] θ1'[t] θ2'[t] + l2^2 m2 θ2'[t]^2)
```

求平衡位形

```
In[120]:= eq1 = {D[V, θ1[t]] == 0, D[V, θ2[t]] == 0} /. {θ1[t] → θ1, θ2[t] → θ2} // Simplify;
                 偏导             偏导                                                 化简

In[121]:= Solve[eq1, {θ1, θ2}, Assumptions → -π/2 < θ1 < π/2 && -π/2 < θ2 < π/2]
          解方程                 假设

Out[121]= {{θ1 → 0, θ2 → 0}}
```

在平衡位形附近作微扰展开，读出二次拉格朗日量,G 与 W 矩阵

```
In[126]:= Series[L, {θ1[t], 0, 2}, {θ2[t], 0, 2}] // Simplify
          级数                                       化简

Out[126]= (1/2 (-2 g (l2 m2 + l1 (m1 + m2)) + l1^2 (m1 + m2) θ1'[t]^2 + 2 l1 l2 m2 θ1'[t] θ2'[t] + l2^2 m2 θ2'[t]^2) +
             1/2 l2 m2 (g - l1 θ1'[t] θ2'[t]) θ2[t]^2 + O[θ2[t]]^3) + (l1 l2 m2 θ1'[t] θ2'[t] θ2[t] + O[θ2[t]]^3) θ1[t] +
           (1/2 l1 (g (m1 + m2) - l2 m2 θ1'[t] θ2'[t]) + 1/4 l1 l2 m2 θ1'[t] θ2'[t] θ2[t]^2 + O[θ2[t]]^3) θ1[t]^2 +
           O[θ1[t]]^3

In[132]:= L2 = 1/2 (l1^2 (m1 + m2) θ1'[t]^2 + 2 l1 l2 m2 θ1'[t] θ2'[t] + l2^2 m2 θ2'[t]^2) + 1/2 l2 m2 g θ2[t]^2 +
            1/2 l1 g (m1 + m2) θ1[t]^2;

In[135]:= G = {{l1^2 (m1 + m2) , l1 l2 m2}, {l1 l2 m2, l2^2 m2 }}; W = {{l1 g (m1 + m2), 0}, {0, l2 m2 g}};

In[140]:= Print["G= ", G // MatrixForm, "     W= ", W // MatrixForm]
          打印              矩阵格式                   矩阵格式

          G= ( l1^2 (m1 + m2)   l1 l2 m2 )      W= ( g l1 (m1 + m2)    0       )
             ( l1 l2 m2         l2^2 m2  )         ( 0                 g l2 m2 )
```

求解 $\det(W-w^2 G)==0$ 等价于 求解 G^{-1} W的本征值与本征矢

```
In[150]:= freq = Eigenvalues[Inverse[G].W] // Simplify
                 特征值        逆              化简
```

w_1

w_2

```
Out[150]= {1/(2 l1 l2 m1) g (l2 m1 + l2 m2 + l1 (m1 + m2) - Sqrt[(m1 + m2) (2 l1 l2 (-m1 + m2) + l1^2 (m1 + m2) + l2^2 (m1 + m2))]),
           1/(2 l1 l2 m1) g (l2 m1 + l2 m2 + l1 (m1 + m2) + Sqrt[(m1 + m2) (2 l1 l2 (-m1 + m2) + l1^2 (m1 + m2) + l2^2 (m1 + m2))])}

In[151]:= vec = Eigenvectors[Inverse[G].W] // Simplify
                特征向量        逆              化简
```

$\vec{A}_1$

$\vec{A}_2$

```
Out[151]= {{(-l2 m1 - l2 m2 + l1 (m1 + m2) + Sqrt[(m1 + m2) (2 l1 l2 (-m1 + m2) + l1^2 (m1 + m2) + l2^2 (m1 + m2))])/(2 l1 (m1 + m2)), 1},
           {-(l2 m1 + l2 m2 - l1 (m1 + m2) + Sqrt[(m1 + m2) (2 l1 l2 (-m1 + m2) + l1^2 (m1 + m2) + l2^2 (m1 + m2))])/(2 l1 (m1 + m2)), 1}}
```

特殊情况： m1=m2=m , l1=l2=l

```
In[159]:= specialize = {m1 → m, m2 → m, l1 → l, l2 → l}; cons = {m > 0, l > 0};

In[160]:= Simplify[freq /. specialize, cons]
          化简

Out[160]= {-((-2 + √2) g)/l, ((2 + √2) g)/l}
```

$\{w_1, w_2\}$

```
In[161]:= Simplify[vec /. specialize, cons]
          化简

Out[161]= {{1/√2, 1}, {-1/√2, 1}}
```

$\{\vec{A}_1, \vec{A}_2\}$

转动理论要点：（以D维欧氏空间为背景）

1. 考虑一坐标变换：$x^i \to \tilde{x}^i = \tilde{x}^i(\vec{x})$

转动要求保度规. i.e. $ds^2 = \delta_{ij} dx^i dx^j = \delta_{kl} d\tilde{x}^k d\tilde{x}^l = \delta_{kl} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} dx^i dx^j$

$\Rightarrow$ 转动条件：$\delta_{kl} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} = \delta_{ij} \Leftrightarrow \delta_{kl} R^k{}_i R^l{}_j = \delta_{ij} \Leftrightarrow R^T I R = I \Leftrightarrow R^T = R^{-1}$.

记 $\frac{\partial \tilde{x}^i}{\partial x^j} \equiv R^i{}_j \Leftrightarrow \frac{\partial x^i}{\partial \tilde{x}^j} = (R^{-1})^i{}_j$，则 R 为正交矩阵，$\det R = \pm 1$.

- 正交归一基间的变换就是转动.

$\langle \vec{e}_i | \vec{e}_j \rangle = \delta_{ij}$. 设 $\vec{e}'_i = R^j{}_i \vec{e}_j$. $\langle \vec{e}'_i | \vec{e}'_j \rangle = \delta_{ij}$，则 $\langle \vec{e}'_i | \vec{e}'_j \rangle = \langle R^k{}_i \vec{e}_k | R^l{}_j \vec{e}_l \rangle = R^k{}_i R^l{}_j \delta_{kl} = \delta_{ij}$. $\Leftrightarrow R^T I R = I$ 正是转动条件.

2 无穷小转动：记 $R = I + \Phi$，Φ 为一无穷小矩阵

由转动条件：$I = R^T R = (I + \Phi^T)(I + \Phi) \simeq I + \Phi^T + \Phi$. 得 $\Phi^T = -\Phi$，i.e. 转动要求 Φ 反对称.

3. 转动群 & 李代数.

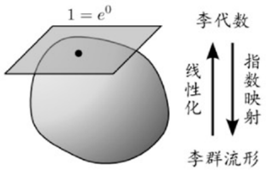


图 11.7: 李群与李代数。

$SO(D) \equiv \{ R \mid \det R = +1, R$ 为D阶正交矩阵$\}$. 同时为流形，为一个李群.

$\dim SO(D) = D^2 - \underset{\substack{\uparrow \\ R^T R \text{ 对称，其给出 } \frac{1}{2}D(D+1) \text{ 约束}}}{\frac{1}{2}D(D+1)} = \frac{1}{2}D(D-1)$. 当 $D=3$ 时，$\dim SO(D) = D = 3$.

$so(D) \equiv \{ \Phi \mid \Phi$ 为D阶实反对称矩阵$\}$，$so(D)$ 在矩阵加法下为线性空间. $\dim so(D) = \frac{1}{2} D(D-1) = \dim SO(D)$.

可取 $\frac{1}{2}D(D-1)$ 个独立的 $\{J_\alpha\}$ 为基，$\forall \Phi \in so(D)$，$\Phi = \sum_{\alpha=1}^{\frac{1}{2}D(D-1)} \phi^\alpha J_\alpha$. J_α ~ 无穷小转动的生成元.

$D=2$：$\dim so(D) = 1$. 可取 $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\Phi_{(2)} = \phi J$. $R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = e^{\phi J} = \lim_{n\to\infty} (1 + \frac{\phi}{n} J)^n$

$D=3$：$\dim so(D) = 3$. $\forall \Phi_{(3)}$ 可写作 $\begin{bmatrix} 0 & -\phi^3 & \phi^2 \\ \phi^3 & 0 & -\phi^1 \\ -\phi^2 & \phi^1 & 0 \end{bmatrix}$ 记：$1 \to 2 \to 3 \to 1$

取 $J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. $J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$，$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

def 对易 $[A, B] = AB - BA$. 3维中 $[J_i, J_j] = \sum_{k=1}^{3} \underset{\substack{\uparrow \\ \text{结构常数. (Hassani. Ch3)}}}{\underline{\epsilon^k{}_{ij}}} J_k$. 门不对易 ~ 3D转动不可交换

绕 x, y, z 轴的有限转动可分别记作：

$R_1(\theta) = e^{\theta J_1} = \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$. $R_2(\theta) = e^{\theta J_2} = \begin{pmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{pmatrix}$. $R_3(\theta) = e^{\theta J_3} = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$.

4. 角速度

▷ $\Omega \equiv (\Omega_i{}^j)$，$R \equiv (R_i{}^j)$

：考虑一依赖于单参数 t 的正交归一基的变换：$\vec{e}_i = \vec{e}_i(t)$.

经过 dt 后，$\vec{e}_i(t) \to \vec{e}_i(t+dt) = (\delta_i{}^j + \Phi_i{}^j(t)) \vec{e}_j(t) \equiv \vec{e}_i(t) + dt \underbrace{\Omega^j{}_i(t) \vec{e}_j(t)}_{\frac{d\vec{e}_i(t)}{dt} = \Omega^j{}_i \vec{e}_j = -\Omega_i{}^j \vec{e}_j}$. i.e. 记 $\Phi_i{}^j(t) = dt\, \Omega^j{}_i(t)$.

有 $\Omega^T = -\Omega$. i.e. $\Omega_i{}^j = -\Omega^j{}_i$.

▷ 考虑两组基，$\{\bar{e}_i\}$ 固定，$\{e_i(t)\}$ 在转动. 设 $e_i(t) = R_i{}^j(t) \bar{e}_j$，$R(t)$ 为转动基相对固定基的转动矩阵.

$\Updownarrow$ $\vec{e}_i = R \vec{\bar{e}}_j$，$R^T \vec{e}_i = \vec{\bar{e}}_j \Leftrightarrow R^j{}_i e_j = \bar{e}_i$

于是 $\frac{de_i(t)}{dt} = \frac{dR_i{}^k}{dt} \bar{e}_k = \underbrace{\frac{dR_i{}^k}{dt} R^j{}_k}_{\equiv \Omega^j{}_i} e_j$

$\Rightarrow \Omega_i{}^j = -\frac{dR_i{}^k}{dt} R^j{}_k = R_i{}^k \frac{dR^j{}_k}{dt} \Leftrightarrow \Omega = -\frac{dR}{dt} R^T = R \frac{dR^T}{dt}$.

转动系中，位矢 $\vec{x} = x^i \vec{e}_i$，速度 $\dot{\vec{x}} = \dot{x}^i \vec{e}_i + x^i \dot{\vec{e}}_i = (\dot{x}^i + \Omega^i{}_j x^j) \vec{e}_i = \vec{v}_0 + \vec{\omega} \times \vec{x}$. ps: $\dot{\vec{v}}_0 = (v_0^i \vec{e}_i)' = \dot{v}_0^i \vec{e}_i + v_0^i \vec{\omega} \times \vec{e}_i = \vec{a}_0 + \vec{\omega} \times \vec{v}_0$.

加速度 $\vec{a} = \ddot{\vec{x}} = (\ddot{x}^i + \underset{\substack{\uparrow \\ \text{非匀速转动}}}{\dot{\Omega}^i{}_j x^j} + \underset{\substack{\uparrow \\ \text{科里奥利效应}}}{2\Omega^i{}_j \dot{x}^j} + \underset{\substack{\uparrow \\ \text{离心效应}}}{\Omega^i{}_k \Omega^k{}_j x^j}) \vec{e}_i = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{x} + 2\vec{\omega} \times \vec{v}_0 + \vec{\omega} \times (\vec{\omega} \times \vec{x})$

dim=3 时，$\Omega = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix} \equiv \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3$. $\{J_1, J_2, J_3; [\,,\,]\} \cong \{\omega_1, \omega_2, \omega_3; \times\}$. $\Omega_{ji} = \epsilon_{ijk} \omega^k$

$\dot{\vec{e}}_i = \Omega^j{}_i \vec{e}_j \underset{\substack{\uparrow \\ 3D}}{=} \epsilon_{ijk} \omega^k \vec{e}_j = \omega^k \vec{e}_k \times \vec{e}_i = \vec{\omega} \times \vec{e}_i$.

转动理论的结论：若动基矢$\{\vec{e}_i\}$相对于固定基$\{\vec{e}_i\}$以$\vec{\omega}$转动，则转动系中$\vec{v}=\vec{v}_0+\vec{\omega}\times\vec{e}_i$，其中$\vec{v}_0=\dot{x}^i\vec{e}_i$.

有$\vec{a}=\vec{a}_0+\dot{\vec{\omega}}\times\vec{x}+2\vec{\omega}\times\vec{v}_0+\vec{\omega}\times(\vec{\omega}\times\vec{x})$.

刚体转动要点：

Euler thm：定点转动$\Leftrightarrow$绕过该点的轴空轴转动.

$\downarrow$ $R\vec{n}=\vec{n}$，$\vec{n}$为轴向

刚体：本体系$\{\vec{e}_i(t)\}$，空间系$\{\vec{\bar{e}}_i\}$

任一点$\vec{x}(t)=x^i\vec{e}_i(t)=x^i(t)\vec{\bar{e}}_i$.

$\vec{e}_i(t)=R_i{}^j(t)\,\vec{\bar{e}}_j$

$\forall\ R,\ R=R^{(3)}(\phi_3)R^{(2)}(\phi_2)R^{(1)}(\phi_1)$

- 欧拉角：$R(\phi,\theta,\psi)=R_3(-\psi)R_1(-\theta)R_3(-\phi)$

$\Omega=\frac{dR}{dt}R^T$也可以欧拉角表示，进而得到$\vec{\omega}$.

✿ 惯量张量

设刚体质量分布为$\rho(\vec{x})$，$\vec{x}=x^i\vec{e}_i$为本体系中某点位矢.

则刚体定点转动动能为：
$$
\begin{aligned}
T&=\frac{1}{2}\int d^3x\,\rho(\vec{x})\vec{v}^2=\frac{1}{2}\int d^3x\,\rho(\vec{x})(\vec{\omega}\times\vec{x})^2\\
&=\frac{1}{2}\int d^3x\,\rho(\vec{x})\,\epsilon_{ijk}\,\omega^i x^j\,\epsilon_{mnl}\,\omega^m x^n\,\delta_{kl}\\
&=\frac{1}{2}\int d^3x\,\rho(\vec{x})(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\,\omega^i x^j\omega^m x^n\\
&=\frac{1}{2}\int d^3x\,\rho(\vec{x})(x^j x_j\delta_{im}\omega^i\omega^m-x_i x_m\omega^i\omega^m)\\
&=\frac{1}{2}\underline{\int d^3x\,\rho(\vec{x})(\vec{x}^2\delta_{ij}-x_i x_j)}\,\omega^i\omega^j\\
&\equiv\frac{1}{2}\underline{I_{ij}}\,\omega^i\omega^j
\end{aligned}
$$

i.e. $T=\frac{1}{2}I_{ij}\omega^i\omega^j$，$I_{ij}=\int d^3x\,\rho(\vec{x})(\vec{x}^2\delta_{ij}-x_i x_j)$. ← 可以记一下.

矩阵形式为：$T=\frac{1}{2}\vec{\omega}^T I\vec{\omega}$，$I=\int d^3x\begin{pmatrix}y^2+z^2&-xy&-xz\\-xy&x^2+z^2&-yz\\-xz&-yz&x^2+y^2\end{pmatrix}\rho(\vec{x})$.

三条惯量主轴.

$\uparrow$

求出I正交归一的本征矢$R^T=\{\vec{e}_1\,\vec{e}_2\,\vec{e}_3\}$ s.t. $I\vec{e}_i=I_i\vec{e}_i$

则$RIR^T=\begin{pmatrix}I_1&&\\&I_2&\\&&I_3\end{pmatrix}$是对角的.

若基矢作变换$\vec{\tilde{e}}_i=R_i{}^j\vec{e}_j$，则$\omega^i=R_i{}^j\tilde{\omega}_j$，$T=\frac{1}{2}\underline{\tilde{I}_{ij}}\tilde{\omega}^i\tilde{\omega}^j=\frac{1}{2}I_{ij}\omega^i\omega^j=\frac{1}{2}\underline{I_{kl}R_i{}^k R_j{}^l}\,\omega^i\omega^j$

$\Updownarrow$

记$(R_i{}^j)\equiv R$　$\vec{\omega}=R\vec{\tilde{\omega}}$

i.e. $\tilde{I}=RIR^T$.

- 平行轴 thm：$I_{ij}^{(P)}=I_{ij}^{(C)}+M(\vec{r}^2\delta_{ij}-r_i r_j)$.（质心为C，$\vec{r}$为P点相对C的位矢）

- 角动量：$\vec{J}=\int d^3x\,\rho(\vec{x})\,\vec{x}\times\vec{v}=\int d^3x\,\rho(\vec{x})\,\vec{x}\times(\vec{\omega}\times\vec{x})\Rightarrow J_i=I_{ij}\omega^j$

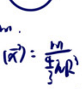
$\Updownarrow$

$p_i=m\delta_{ij}v^j$

eg. $I_杆$，$I_球$

总重m，均匀

$$I_杆=\int_{-\frac{l}{2}}^{\frac{l}{2}}dz\int dx\int dy\,\delta(x)\delta(y)\frac{m}{l}\begin{pmatrix}y^2+z^2&-xy&-xz\\-xy&x^2+z^2&-yz\\-xz&-yz&x^2+y^2\end{pmatrix}=\frac{m}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}dz\begin{pmatrix}z^2&0&0\\0&z^2&0\\0&0&0\end{pmatrix}=\frac{ml^2}{12}\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}$$



重m.

$\rho(\vec{x})=\frac{m}{\frac{4}{3}\pi R^3}$

$\begin{cases}x=r\sin\theta\cos\phi\\y=r\sin\theta\sin\phi\\z=r\cos\theta\end{cases}$

$$
\begin{aligned}
I_球&=\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^R dr\,\frac{m}{\frac{4}{3}\pi R^3}r^2\begin{pmatrix}\sin^2\theta\sin^2\phi+\cos^2\theta&-\sin^2\theta\cos\phi\sin\phi&-\sin\theta\cos\theta\cos\phi\\-\sin^2\theta\cos\phi\sin\phi&\sin^2\theta\cos^2\phi+\cos^2\theta&-\sin\theta\cos\theta\sin\phi\\-\sin\theta\cos\theta\cos\phi&-\sin\theta\cos\theta\sin\phi&\sin^2\theta\end{pmatrix}r^2\sin\theta\\
&=\frac{m}{\frac{4}{3}\pi R^3}\cdot\frac{1}{5}R^5\cdot\int_0^{\pi}d\theta\begin{pmatrix}\pi\sin^2\theta+2\pi\cos^2\theta&&\\&\pi\sin^2\theta+2\pi\cos^2\theta&\\&&2\pi\sin^2\theta\end{pmatrix}\sin\theta\\
&=\frac{3mR^2}{20\pi}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\cdot\frac{8}{3}\pi=\frac{2mR^2}{5}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}.
\end{aligned}
$$

$$
\because\begin{cases}\int_0^{\pi}\sin^3\theta\,d\theta=-\int_0^{\pi}(1-\cos^2\theta)\,d(\cos\theta)=\frac{1}{3}\cos^3\theta-\cos\theta\Big|_0^{\pi}=\frac{4}{3}\\\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta=-\int_0^{\pi}\cos^2\theta\,d\cos\theta=-\frac{1}{3}\cos^3\theta\Big|_0^{\pi}=\frac{2}{3}\end{cases}
$$

哈密顿力学要点* ⇒ $[q^a,q^b]=0$, $[p_a,p_b]=0$, $[q^a,p_b]=\delta^a{}_b$. $a,b=1,\dots,s$.

$\begin{cases}\text{正则方程}\\ \text{泊松括号}\\ \text{哈密顿—雅可比方程.}\end{cases}$

哈密顿量: $H(t,\vec q,\vec p)\equiv p_a\dot q^a-L$, $\dot q^a$ 看作 $\dot q^a=\dot q^a(t,\vec q,\vec p)$.

$$\begin{cases}dH=\frac{\partial H}{\partial t}dt+\frac{\partial H}{\partial q^a}dq^a+\frac{\partial H}{\partial p_a}dp_a\\ dH\underset{\text{E-L eqn}}{=}-\frac{\partial L}{\partial t}dt-\dot p_a dq^a+\dot q^a dp_a\end{cases}$$ 对比可得 正则方程, 及 $\frac{dH}{dt}=-\frac{\partial L}{\partial t}=\frac{\partial H}{\partial t}$.

至此可以总结一下，由系统的拉格朗日量 $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 出发，得到其哈密顿正则方程的步骤：

1. 计算共轭动量 $p_a=\frac{\partial L}{\partial\dot q^a}\equiv p_a(t,\boldsymbol{q},\dot{\boldsymbol{q}})$;
2. 从共轭动量的表达式 $p_a=p_a(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 中反解出广义速度，作为广义坐标和广义动量的函数 $\dot q^a=\dot q^a(t,\boldsymbol{q},\boldsymbol{p})$;
3. 将 $\dot q^a=\dot q^a(t,\boldsymbol{q},\boldsymbol{p})$ 带入哈密顿量的定义 $H=p_a\dot q^a-L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$，将所有的广义速度换成广义坐标和广义动量的函数，得到哈密顿量 $H=H(t,\boldsymbol{q},\boldsymbol{p})$;
4. 对哈密顿量求导，得到哈密顿正则方程 (13.19)。 $\begin{cases}\dot q^a=\frac{\partial H}{\partial p_a}\\ \dot p_a=-\frac{\partial H}{\partial q^a}\end{cases}$

$\begin{cases}\dot q^a=u^a(t,\vec q,\vec p)\\ \dot p_a=v_a(t,\vec q,\vec p)\end{cases}$ 描述哈密顿系统的条件:

$$\left.\begin{matrix}\frac{\partial\dot q^a}{\partial q^b}\equiv\frac{\partial u^a}{\partial q^b}=\frac{\partial^2H}{\partial q^b\partial p_a}, & \frac{\partial\dot q^a}{\partial p_b}\equiv\frac{\partial u^a}{\partial p_b}=\frac{\partial^2H}{\partial p_b\partial p_a}\\ \frac{\partial\dot p_a}{\partial q^b}\equiv\frac{\partial v_a}{\partial q^b}=-\frac{\partial^2H}{\partial q^b\partial q^a}, & \frac{\partial\dot p_a}{\partial p_b}\equiv\frac{\partial v_a}{\partial p_b}=-\frac{\partial^2H}{\partial p_b\partial q^a}\end{matrix}\right\}\Rightarrow\underline{\frac{\partial u^a}{\partial q^b}=-\frac{\partial v^a}{\partial p_b}},\ \underline{\frac{\partial u^a}{\partial p^b}=\frac{\partial u^b}{\partial p^a}},\ \underline{\frac{\partial v^a}{\partial q^b}=\frac{\partial v^b}{\partial q^a}}.$$

· 劳斯方法: 揉合怪… 用于处理循环坐标

· 泊松括号, 辛几何　　重记 $\{q^1,\dots,q^s;p_1,\dots,p_s\}$ 为 $\{\xi^1,\dots,\xi^{2s}\}$, 2s个相空间坐标.

$\underset{(2s)}{\omega^{\alpha\beta}}\equiv\begin{pmatrix}0_s & I_s\\ -I_s & 0\end{pmatrix}\equiv\omega^{-1}$. 正则方程 ⇔ $\dot\xi^\alpha=\omega^{\alpha\beta}\frac{\partial H}{\partial\xi^\beta}$.　　如 s=1时, $\begin{pmatrix}\dot q\\ \dot p\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{pmatrix}$

则 其逆 $\omega=\omega_{\alpha\beta}=\begin{pmatrix}0&-I\\ I&0\end{pmatrix}$, 反对称. $\det\omega=1$. $\omega^T=-\omega=\omega^{-1}$.

以 $\omega^{\alpha\beta}$ 定义辛内积: $\langle X|Y\rangle=\omega^{\alpha\beta}X_\alpha Y_\beta$.

Poisson 括号: $[f,g]:=\langle\nabla f|\nabla g\rangle=\omega^{\alpha\beta}\frac{\partial f}{\partial\xi^\alpha}\frac{\partial g}{\partial\xi^\beta}$

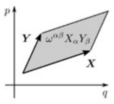


图 14.3: 辛内积的几何意义——-面积。

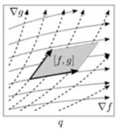


图 14.4: 泊松括号的几何意义。

· 具体形式: $[f,g]=\sum_{\alpha=1}^{2s}\sum_{\beta=1}^{2s}\omega^{\alpha\beta}\frac{\partial f}{\partial\xi^\alpha}\frac{\partial g}{\partial\xi^\beta}=\Big(\underbrace{\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}}_{=0}+\underbrace{\sum_{\alpha=1}^{s}\sum_{\beta=s+1}^{2s}}_{=\sum_{a,b=1}^{s}\delta^{ab}\frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_b}}+\underset{!!}{\sum_{\alpha=s+1}^{2s}\sum_{\beta=1}^{s}}+\underbrace{\sum_{\alpha=s+1}^{2s}\sum_{\beta=s+1}^{2s}}_{=0}\Big)\omega^{\alpha\beta}\frac{\partial f}{\partial\xi^\alpha}\frac{\partial g}{\partial\xi^\beta}$

$\Rightarrow\underline{[f,g]=\frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a}-\frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}}$

▷ properties:
① Antisymmetric: 由 $\omega^{\alpha\beta}=-\omega^{\beta\alpha}$, 必有 $[f,g]=-[g,f]\overset{\text{推论}}{\Rightarrow}[f,f]=0$.
② bilinear: $[af+bg,h]=a[f,h]+b[g,h]$; $[f,ag+bh]=a[f,g]+b[f,h]$.
③ Leibniz's rule: $[fg,h]=f[g,h]+[f,h]g$; $[f,gh]=g[f,h]+[f,g]h$

意义: 可把 $[f,\bullet]$ 或 $[\bullet,f]$ 视作对 • 作用的某种"求导"运算.

④ Jacobi's identity: $[[f,g],h]+[[g,h],f]+[[h,f],g]=0$.
$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$.
⑤ Chain rule: $[F(f),g]=\frac{\partial F}{\partial f}[f,g]$, $[f,G(g)]=[f,g]\frac{\partial G}{\partial g}$
⑥ $\frac{\partial[f,g]}{\partial\lambda}=[\frac{\partial f}{\partial\lambda},g]+[f,\frac{\partial g}{\partial\lambda}]$. ⑦ Poisson thm: $\frac{d[f,g]}{dt}=[\frac{df}{dt},g]+[f,\frac{dg}{dt}]$. ⇒ 推论: 若 f, g 为两运动常数, 则 [f,g] 也为运动常数.

▷ fundamental Poisson brackets:

· 考虑 $[\xi^\alpha,f]=\omega^{\gamma\beta}\frac{\partial\xi^\alpha}{\partial\xi^\gamma}\frac{\partial f}{\partial\xi^\beta}=\omega^{\gamma\beta}\delta^\alpha_\gamma\frac{\partial f}{\partial\xi^\beta}=\omega^{\alpha\beta}\frac{\partial f}{\partial\xi^\beta}$. ⇒ $\underline{[\xi^\alpha,\bullet]=\omega^{\alpha\beta}\frac{\partial\bullet}{\partial\xi^\beta}}$.

⇒ $\begin{pmatrix}[\vec q,\bullet]\\ [\vec p,\bullet]\end{pmatrix}=\begin{pmatrix}0_s&1_s\\-1_s&0_s\end{pmatrix}\begin{pmatrix}\frac{\partial\bullet}{\partial\vec q}\\ \frac{\partial\bullet}{\partial\vec p}\end{pmatrix}=\begin{pmatrix}\frac{\partial\bullet}{\partial\vec p}\\ -\frac{\partial\bullet}{\partial\vec q}\end{pmatrix}$, i.e. $[q^a,\bullet]=\frac{\partial\bullet}{\partial p_a}$, $[p_a,\bullet]=-\frac{\partial\bullet}{\partial q^a}$, $a=1,\dots,s$.　<u>泊松括号～求偏导</u>

⇒ ∀f, g, $[f,g]=\frac{\partial f}{\partial\xi^\alpha}[\xi^\alpha,\xi^\beta]\frac{\partial g}{\partial\xi^\beta}$. 由 [,] 定义, $[\xi^\alpha,\xi^\beta]=\omega^{\alpha\beta}$, $\alpha,\beta=1,\dots,2s$.

* ⇒ $[q^a,q^b]=0$, $[p_a,p_b]=0$, $[q^a,p_b]=\delta^a{}_b$. $a,b=1,\dots,s$.

▷ 物理量的演化 $f=f(t,\vec q,\vec p)=f(t,\vec\xi)$

· $\dot f=\frac{\partial f}{\partial t}+\dot\xi^\alpha\frac{\partial f}{\partial\xi^\alpha}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial\xi^\alpha}\omega^{\alpha\beta}\frac{\partial H}{\partial\xi^\beta}=\frac{\partial f}{\partial t}+[f,H]$. → 泊松括号形式的动力学方程

· 运动常数: $\dot f=\frac{\partial f}{\partial t}+[f,H]=0$

若f不显含时间, 则 $\dot f=[f,H]=0$ ⟺ f 与 H 对易.

· 对于 q^a 为循环坐标, $\dot p_a=[p_a,H]=-\frac{\partial H}{\partial q^a}=0$, 自然有$p_a$为运动常数.

易 $\frac{dH}{dt}=\frac{\partial H}{\partial t}+[H,H]=\frac{\partial H}{\partial t}$. 若H不显含时间, 则 $\dot H=0$, H为运动常数.

· 角动量の…

$$J^i=\epsilon^{ijk}x_jP_k\quad\Rightarrow\quad\epsilon_{ijk}J^k=\epsilon_{ijk}\epsilon^{kmn}x_mP_n=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})x_mP_n=x_iP_j-x_jP_i$$

· $[J^i,x^j]=[\epsilon^{ikl}x_kP_l,x^j]=\epsilon^{ikl}x_k[P_l,x^j]=-\epsilon^{ikl}x_k\delta^j{}_l=\epsilon^{ijk}x_k$

$[J^i,p^j]=[\epsilon^{ikl}x_kP_l,p^j]=\epsilon^{ikl}P_l[x_k,p^j]=\epsilon^{ikl}P_l\delta_k^j=\epsilon^{ijk}P_k$

$$\begin{aligned}[J^i,J^j]&=[J^i,\epsilon^{jkl}x_kP_l]=\epsilon^{jkl}[J^i,x_k]P_l+\epsilon^{jkl}x_k[J^i,P_l]\\&=\epsilon^{jkl}\epsilon^i{}_{km}x^mP_l+\epsilon^{jkl}\epsilon^i{}_{lm}x_kP^m\\&=(\delta^l{}_m\delta^{ij}-\delta^{li}\delta^j{}_m)x^mP_l+(\delta^j{}_m\delta^{ki}-\delta^{ji}\delta^k{}_m)x_kP^m\\&=\delta^{ij}\cancel{x^m}P_m-x^jp^i+x^ip^j-\delta^{ij}\cancel{x_m}P^m=x^ip^j-x^jp^i\end{aligned}$$

i.e. $\underline{[J^i,J^j]=\epsilon^{ijk}J_k}$. ⟺ $[J^1,J^2]=J^3$, $[J^2,J^3]=J^1$, $[J^3,J^1]=J^2$. ≃ dim 3 转动生成元の李代数

↑这违背了基本泊松括号, 故角动量分量不能同时为正则坐标/动量.

f是标量嘛.

· 由于 $\vec x$ 与 $\vec p$ 间仅存3种缩并, $\vec x^2=x_ix^i$, $\vec p^2=P_iP^i$, $\vec x\cdot\vec p=x^iP_i$. 故 $\forall f=f(t,\vec x,\vec p)$, 有 $f=f(t,\vec x^2,\vec p^2,\vec x\vec p)$.

· 有 $[J^i,\vec x^2]=[J^i,\vec p^2]=[J^i,\vec x\cdot\vec p]=0\Rightarrow[J^i,f]=0\Rightarrow[J^i,\vec J^2]=0$; $[J^i,V^j]=\epsilon^{ijk}V_k$, $\vec V$为相空间上任意矢量

Pf: $[J^i,\vec x^2]=[J^i,\delta_{\mu\nu}x^\mu x^\nu]=2\delta_{\mu\nu}x^\mu[J^i,x^\nu]=2\delta_{\mu\nu}x^\mu\epsilon^{i\nu k}x_k=2\epsilon^{ijk}x_jx_k=0$. $[J^i,\vec p^2]$同理.

$[J^i,\vec x\cdot\vec p]=[J^i,\delta_{\mu\nu}x^\mu p^\nu]=\delta_{\mu\nu}x^\mu[J^i,p^\nu]+\delta_{\mu\nu}p^\nu[J^i,x^\mu]=\epsilon^{i\nu k}x_\nu p_k+\epsilon^{i\nu k}p_\nu x_k=0$. □

Pf: $[J^i,f]=[J^i,f(t,\vec x^2,\vec p^2,\vec x\cdot\vec p)]=\frac{\partial f}{\partial(\vec x^2)}[J^i,\vec x^2]+\frac{\partial f}{\partial(\vec p^2)}[J^i,\vec p^2]+\frac{\partial f}{\partial(\vec x\cdot\vec p)}[J^i,\vec x\cdot\vec p]=0$. 自然可取 $f=\vec J^2=\vec J\cdot\vec J$.

而 相空间中任意矢量 只能为 $\vec x,\vec p,\vec J$ 的线性组合. 不妨写作 $V^i=fx^i+gp^i+hJ^i$. 故 $[J^i,V^j]=f[J^i,x^j]+g[J^i,p^j]+h[J^i,J^j]=\epsilon^{ijk}V_k$

· 正则变换: 保辛矩阵 ⟺ 保泊松括号 ⟺ 保哈密顿正则方程.

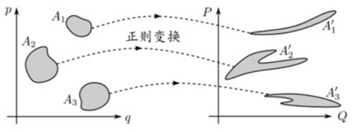


图 15.1: 正则变换保辛矩阵不变，从而保面积不变。

记相空间坐标为 $\{\xi^\alpha\}$, 变换后为 $\{\tilde\xi^\alpha\}$.

正则变换条件: $\omega^{\rho\sigma}\frac{\partial\tilde\xi^\alpha}{\partial\xi^\rho}\frac{\partial\tilde\xi^\beta}{\partial\xi^\sigma}=\omega^{\alpha\beta}$ ⟺ $\omega^{\rho\sigma}\frac{\partial\xi^\alpha}{\partial\tilde\xi^\rho}\frac{\partial\xi^\beta}{\partial\tilde\xi^\sigma}=\omega^{\alpha\beta}$.

⟺ $[\tilde\xi^\alpha,\tilde\xi^\beta]_\xi=\omega^{\alpha\beta}$ ⟺ $[\xi^\alpha,\xi^\beta]_{\tilde\xi}=\omega^{\alpha\beta}$.

相空间 坐标$\{\vec q,\vec p\}$下, $H=H(t,\vec q,\vec p)$. $\begin{cases}\dot q^a=\frac{\partial H}{\partial p_a}\\ \dot p_a=-\frac{\partial H}{\partial q^a}\end{cases}$

(正则)变换

$\{\vec Q,\vec P\}$, 变换后Hamiltonian $K=K(t,\vec Q,\vec P)$ $\begin{cases}\dot Q^a=\frac{\partial K}{\partial P_a}\\ \dot P_a=\frac{\partial K}{\partial Q^a}\end{cases}$ (保 Hamilton's canonical eqn 的要求)

一般 $K(t,\vec Q,\vec P)$与$H(t,\vec q,\vec p)$ 数值与函数形式 均不同

($S=\lambda\tilde S$ 为准正则 (canonoid). 不满足基本泊松括号)

由于 哈密顿正则方程可由 相空间的变分原理导出. 故上述要求 ⟺ $\begin{cases}\delta S(\vec q,\vec p)=0\\ \delta\tilde S(\vec Q,\vec P)=0\end{cases}$ 同时成立 ⟹ $S=\tilde S$

(由拉氏量): $p_a\dot q^a-H(t,\vec q,\vec p)=P_a\,d\,Q^a-K(t,\vec Q,\vec P)+\frac{dF}{dt}$. F 取决于正则变换关系

⟹ $dF=p_adq^a-P_adQ^a+[K(t,\vec Q,\vec P)-H(t,\vec q,\vec p)]dt$. $\begin{cases}\text{若变换为正则变换, RHS必为全微分}\\ \text{若 }\vec q,\vec Q\text{ 独立, 则 }F=F(t,\vec q,\vec Q).\end{cases}$

· 对于固定时间t, $\delta F=p_a\delta q^a-P_a\delta Q^a$

可直接得到偏导关系: $F=F(t,\vec q,\vec Q)$. $\begin{cases}\frac{\partial F}{\partial q^a}=p_a\quad,a=1,\dots,s\\ \frac{\partial F}{\partial Q^a}=-P_a\\ \frac{\partial F}{\partial t}=K(t,\vec Q,\vec P)-H(t,\vec q,\vec p)\end{cases}$

} 给出了$\{\vec q,\vec p\}$与$\{\vec Q,\vec P\}$的2s个方程.

当且仅当 $\det\left(\frac{\partial^2F}{\partial q^a\partial Q^b}\right)\neq0$ 可显式解出 $Q^a=Q^a(t,\vec q,\vec p)$, $P_a=P_a(t,\vec q,\vec p)$

↘ 新旧 Hamiltonian 关系: $K(t,\vec Q,\vec P)=H(t,\vec q,\vec p)+\frac{\partial F(t,\vec q,\vec Q)}{\partial t}$

该差别全由$\frac{\partial F}{\partial t}$, i.e. 正则变换本身决定! ~正则变换的几何属性 (与具体物理系统无关).

此$F(t,\vec q,\vec Q)$即生成函数.

▷ 四类生成函数:

	$\vec Q$	$\vec P$
$\vec q$	F_1	F_2
$\vec p$	F_3	F_4

F_i 满足的偏导关系可由 dF_i 直接读出.

比如 想知道 $F_2(t,\vec P,\vec q)$的偏导关系

由 $dF_1=p_adq^a-P_adQ^a+(K-H)dt=p_adq^a+Q^adP_a+(K-H)dt-dP_aQ^a$

可知 $d(F_1+P_aQ^a)\equiv dF_2=\underbrace{p_a}_{\frac{\partial F_2}{\partial q^a}}dq^a+\underbrace{Q^a}_{\frac{\partial F_2}{\partial P_a}}dP_a+(K-H)dt$.

eg. $F_2=q^aP_a$

◦单参数正则变换（与生成函数并列的一种生成正则变换的方式）

考虑 无穷小变换 $\xi^\alpha \to \xi^\alpha + \epsilon X^\alpha(\vec\xi)$, $\alpha=1,\dots,2s$. ϵ 为无穷小参数, $X^\alpha = X^\alpha(\vec\xi)$ 为相空间中的矢场.

◦无穷小正则变换: $[\xi^\alpha+\epsilon X^\alpha, \xi^\beta+\epsilon X^\beta] \simeq [\xi^\alpha,\xi^\beta]+\epsilon[\xi^\alpha,X^\beta]+\epsilon[X^\alpha,\xi^\beta] = \omega^{\alpha\beta}+\epsilon\left(\omega^{\alpha\rho}\frac{\partial X^\beta}{\partial\xi^\rho} - \omega^{\beta\rho}\frac{\partial X^\alpha}{\partial \xi^\rho}\right) \underset{\text{要求}}{=} \omega^{\alpha\beta}$

i.e. 要求为 $\underline{\omega^{\alpha\rho}\frac{\partial X^\beta}{\partial\xi^\rho} = \omega^{\beta\rho}\frac{\partial X^\alpha}{\partial\xi^\rho}}$. 若以 q^a, p_a 写出, 记为 $\begin{cases} q^a \to q^a+\epsilon u^a \\ p_a \to p_a+\epsilon v_a \end{cases}$, 则 $\Leftrightarrow$ $\frac{\partial u^b}{\partial p_a} = \frac{\partial u^a}{\partial p_b}$, $\frac{\partial v_b}{\partial q^a}=\frac{\partial v_a}{\partial q^b}$, $\frac{\partial u^a}{\partial q^b} = -\frac{\partial v_b}{\partial p_a}$.

Pf: 以 q^a, p_a 形式写出后, $\begin{cases} [q^a+\epsilon u^a, q^b+\epsilon u^b]=0 \\ [p_a+\epsilon v_a, p_b+\epsilon v_b]=0 \\ [q^a+\epsilon u^a, p_b+\epsilon v_b]=0 \end{cases}$. 化开就好了.

·形式与 H-C eqn 判断条件一致, y?

当 X^α 为相空间上函数 $G(t,\vec\xi)$ 的梯度时, $X^\alpha \equiv \omega^{\alpha\beta}\frac{\partial G}{\partial\xi^\beta} := X^\alpha_G = [\xi^\alpha, G]$. $\alpha=1,\dots,2s$.

此时 $\underline{\text{条件 } \omega^{\alpha\rho}\frac{\partial X^\beta}{\partial \xi^\rho} = \omega^{\beta\rho}\frac{\partial X^\alpha}{\partial\xi^\rho}}$ 可写作: $(\omega^{\alpha\rho}\omega^{\beta\sigma}-\omega^{\beta\rho}\omega^{\alpha\sigma})\frac{\partial^2 G}{\partial\xi^\rho\partial\xi^\sigma}=0$, 天然成立! （考虑 (), [], ...）

这相应的 $X^\alpha_G \equiv \omega^{\alpha\beta}\frac{\partial G}{\partial\xi^\beta} = [\xi^\alpha, G]$ 称作 Hamiltonian vector field

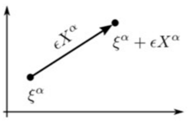


图 15.6: 相空间的无穷小坐标变换。

⇒ 给定相空间上函数 $\overset{\text{生成元}}{G}(t,\vec\xi)$, 便可得一无穷小正则变换.

记作 $\xi^\alpha\to\xi^\alpha+\delta\xi^\alpha$, $\alpha=1,\dots,2s$.

其中 $\delta_G\xi^\alpha = \epsilon[\xi^\alpha,G]\equiv\epsilon X^\alpha_G$.

具体写出即: $\delta q^a=\epsilon[q^a,G]=\epsilon\frac{\partial G}{\partial p_a}$; $\delta p_a = \epsilon[p_a,G] = -\frac{\partial G}{\partial q^a}$.

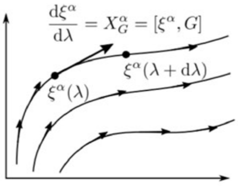


图 15.7: 相流与哈密顿矢量场。

▷ 连续变换:

取参数为 λ. 无穷小参数为 $d\lambda$.

则 无穷小正则变换: $\xi^\alpha(\lambda)\to\xi^\alpha(\lambda)+\delta_G\xi^\alpha = \xi^\alpha(\lambda)+\overbrace{d\lambda[\xi^\alpha,G]}^{d\xi^\alpha}\equiv\xi^\alpha(\lambda+d\lambda)$

⇒ $\frac{d\xi^\alpha(\lambda)}{d\lambda}=[\xi^\alpha,G]\equiv X^\alpha_G$, $\alpha=1,\dots,2s$. 相流的"流速", 正是哈密顿矢量场!

◦与生成函数法的一致性:

考虑 恒等变换 $F_2=q^aP_a$. 则其经无穷小正则变换后有形式 $F_2=q^aP_a+\epsilon W(t,\vec q,\vec P)$.

由 F_2 的条件, $p_a=\frac{\partial F_2}{\partial q^a}=P_a+\epsilon\frac{\partial W}{\partial q^a}$, $Q^a=\frac{\partial F_2}{\partial P_a}=q^a+\epsilon\frac{\partial W}{\partial P_a}$. ($G=\lim_{\epsilon\to0}W(t,\vec q,\vec P)$)

⇒ 当 $\epsilon\to0$, W及其导数可取在 $\vec q,\vec p$ 的值, $\delta q^a=Q^a-q^a\simeq\epsilon\frac{\partial G}{\partial p_a}$, $\delta p_a=P_a-p_a=-\epsilon\frac{\partial G}{\partial q^a}$.

发现此G正是无穷小正则变换の生成元!

演化即正则变换. 生成元为H, 参数为t.

H-C eqn: $\frac{d\xi^\alpha(t)}{dt}=\omega^{\alpha\beta}\frac{\partial H}{\partial\xi^\beta}=[\xi^\alpha,H]\equiv X^\alpha_H$.

对 力学量 $f=f(t,\vec\xi)$, 无穷小时间演化即 $\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial\xi^\alpha}\frac{d\xi^\alpha}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial\xi^\alpha}[\xi^\alpha,H]=\frac{\partial f}{\partial t}+[f,H]$,

从正则变换角度导出力学量演化方程.

- H-C eqn 实际为 正则变换的具体形式.

- 对称性 与生成元.

Def $\hat G\cdot := [\cdot, G]$. 则 力学量 f 的有限正则变换: $f(\lambda)=e^{\lambda\hat G}f(0)$. [对应 相空间坐标 $\vec\xi(0)\to\vec\xi(\lambda)$]

· 考虑 相空间上函数 G. G不含时. 且 $[G,H]=0$.

按照 (15.87)，这个式子可以从两个角度解读：

- $\delta_H G=0$，即视为 G 在 H 所生成的无穷小正则变换下不变。因为哈密顿量 H 生成时间演化，这无非是说力学量 G 在时间演化下不变，即是运动常数。
- $\delta_G H=0$，即视为 H 在 G 所生成的无穷小正则变换下不变。因为哈密顿量包含了系统的全部动力学信息，因此哈密顿量 H 不变，则意味着 G 生成的无穷小正则变换是系统的对称性。

对不含时的力学量 G,

~ G是运动常数 $\Leftrightarrow$ G是对称变换生成元

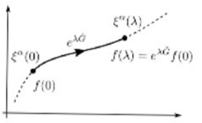


图 15.9: 有限正则变换与指数映射。

哈密顿-雅克比理论

· 时间演化即正则变换 ~ 求解演化 ≡ 寻找一正则变换，将系统状态与一常数联系起来.

Hamilton-Jacobi theory 正是描写如何得到该正则变换、其"生成函数"的理论

▷ 求解 $\{\vec{q}(t),\vec{p}(t)\}$. IC 记作 $\{\vec{\underline{q}},\vec{\underline{p}}\}=\{\underline{q}^1,\ldots,\underline{q}^s,\underline{p}_1,\ldots\underline{p}_s\}$ 为2s个常数.

$$\{\vec{q}(t),\vec{p}(t)\}\xrightleftharpoons[\text{正则变换}]{}\{\vec{\underline{q}},\vec{\underline{p}}\}.$$

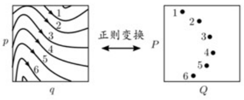


图 16.1: 哈密顿-雅可比理论的几何图像：把相流变成点。

若正则变换使 $\{\vec{q},\vec{p}\}\to\{\vec{Q},\vec{P}\}$. 变换后 2s 个坐标 $\{\vec{Q},\vec{P}\}$ 为常数,

则新 Hamiltonian 应满足 $K=H+\frac{\partial F}{\partial t}=0$.

↑ 不妨再令常数为0.

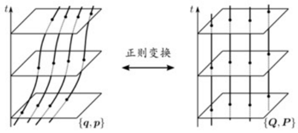


图 16.2: 哈密顿-雅可比理论的几何图像：将时间轨迹拉直。

▷ 习惯取2型生成函数：$F_2=F_2(t,\vec{q},\vec{P})$

要求 $\{\vec{Q},\vec{P}\}\equiv\{\vec{\beta},\vec{\alpha}\}=\{\beta^1,\ldots,\beta^s,\alpha_1,\ldots,\alpha_s\}=const.$ （$\vec{\beta},\vec{\alpha}$ 可为 $\vec{\underline{p}},\vec{\underline{q}}$ 的2s个独立组合）

于是可把生成函数记作：$F_2(t,\vec{q},\vec{P})\equiv S(t,\vec{q},\vec{\alpha})$ ← Hamilton's principal function

S为 $t,\vec{q}$ 的函数，$\vec{\alpha}=\{\alpha_1,\ldots,\alpha_s\}$ 为s个参数. 有作用量量纲.

· 有 $p_a=\frac{\partial S}{\partial q^a}$，$Q^a\equiv\beta^a=\frac{\partial S}{\partial\alpha_a}\equiv const$，$K=H+\frac{\partial S}{\partial t}\equiv0$.

可知S为一解，则 $S+S_0$ 也为一解，$S_0=const$

其中 $H(t,\vec{q},\vec{p})+\frac{\partial S(t,\vec{q},\vec{\alpha})}{\partial t}=0$ 可写作 $\underline{H(t,q^1,\ldots,q^s,\frac{\partial S}{\partial q^1},\ldots,\frac{\partial S}{\partial q^s})+\frac{\partial S}{\partial t}=0}$ ~ Hamilton-Jacobi eqn，关于S的1st PDE

Steps:

可以总结一下利用哈密顿-雅可比方程求解问题的具体步骤：

1. 根据系统的哈密顿量，将其中的广义动量 p_a 替换为 $\frac{\partial S}{\partial q^a}$，按照 (16.12) 写出哈密顿-雅可比方程；
   $H(t,q^1,\ldots,q^s,\frac{\partial S}{\partial q^1},\ldots,\frac{\partial S}{\partial q^s})+\frac{\partial S}{\partial t}=0.$
2. key: 求哈密顿-雅可比方程的完全积分 $S(t,\boldsymbol{q},\boldsymbol{\alpha})$，其中包含 s 个非可加的常数 $\{\alpha_1,\cdots,\alpha_s\}$，并满足非退化条件 (16.14)；$\det\left(\frac{\partial^2S(t,\vec{q},\vec{\alpha})}{\partial q^a\partial\alpha_b}\right)\neq0$. 只须求出S的特解
3. 将常数 $\{\alpha_1,\cdots,\alpha_s\}$ 作为变换后的广义动量，根据正则变换关系 (16.10)，求得 s 个广义坐标 (16.13)；反解 $q^a=q^a(t,\vec{\beta},\vec{\alpha})$
   $\beta^a=\frac{\partial S}{\partial\alpha_a}$ （$Q^a=\frac{\partial F_2}{\partial P_a}$）
4. 根据正则变换关系 (16.9)，并代入 (16.13)，求得 s 个广义动量 (16.15)。
   $p_a=\frac{\partial S}{\partial q^a}$ $p_a=p_a(t,\vec{\beta},\vec{\alpha})=\left.\frac{\partial S(t,\vec{q},\vec{\alpha})}{\partial q^a}\right|_{q^a=q^a(t,\vec{\beta},\vec{\alpha})}$.

怎么解：
{ 可分离变量
 不可 ~ 硬解 PDE. / 换种理论算.

▷ 分离变量法 ~ 含循环座标 / 不含时系统.

若s自由度系统有s-k个循环坐标：（不妨记循环坐标为 $\{q^{k+1},\ldots,q^s\}$，共轭动量 $\{p_{k+1},\ldots,p_s\}\equiv\{\alpha_1,\ldots,\alpha_s\}$.）

只须对这些坐标作恒等变换即可. $F_2=q^iP_i=q^ip_i=q^i\alpha_i$.

由 $p_i=\frac{\partial S}{\partial q^i}=\alpha_i=const$，$i=k+1,\ldots,s$. 故 $S(t,\vec{q})=\underbrace{\bar{S}(t,q^1,\ldots,q^k)}_{非循环}+\underbrace{\alpha_{k+1}q^{k+1}+\ldots+\alpha_sq^s}_{循环}$

~ 约化为求解 $H(t,q^1,\ldots,q^k,\frac{\partial\bar{S}}{\partial q^1},\ldots,\frac{\partial\bar{S}}{\partial q^k};\alpha_{k+1},\ldots,\alpha_s)+\frac{\partial\bar{S}}{\partial t}=0$.

✡ 若H不含时，设 $S(t,\vec{q})$ 具有形式 $S(t,\vec{q})\equiv W(\vec{q})+V(t)$. → Hamilton's characteristic function

两边参数独立，只能同时为常数

于是有 $\frac{\partial S}{\partial q^a}=\frac{\partial W}{\partial q^a}$，代入 H-J eqn 得 $H(\vec{q},\frac{\partial W(\vec{q})}{\partial\vec{q}})=-V'(t)\equiv const=E$. $\to V=-Et$，$S=W(\vec{q})-Et$. 注：如此求解的 $S(t,\vec{q},\vec{\alpha})$ 未必满足非退化条件

$\Leftrightarrow H(q^1,\ldots,q^s,\frac{\partial W}{\partial q^1},\ldots,\frac{\partial W}{\partial q^s})=E=const$ PS：给定E，W的特解含s个积分常数，1个可加，s-1个不可加，记作 $\{\alpha_2,\ldots,\alpha_s\}$. 记 $\alpha_1\equiv E$，则 $W=W(\vec{q},E,\alpha_2,\ldots,\alpha_s)$ 含s个不可加常数.

解出来要看看S是否非退化.

eg. 一维谐振子.

$L=\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2$，$p=\frac{\partial L}{\partial\dot{q}}=m\dot{q}\to\dot{q}=\frac{p}{m}$

$\Rightarrow H(q,p)=p\dot{q}-L=\frac{p^2}{m}-\frac{1}{2}m\frac{p^2}{m^2}+\frac{1}{2}m\omega^2q^2=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$

正则方程：$\begin{cases}\dot{q}=\frac{\partial H}{\partial p}=\frac{p}{m}\\ \dot{p}=\frac{\partial H}{\partial q}=-m\omega^2q\end{cases}$

解1：对 q, p 作变换：$q=\sqrt{\frac{2P}{m\omega}}\sin Q$，$p=\sqrt{2m\omega P}\cos Q$.

则变换后哈密顿量 $K=\omega P\equiv E$. 有 $P=\frac{E}{\omega}$，$\dot{Q}=\frac{\partial K}{P}=\omega\xrightarrow{\text{积}}Q=\omega t+\varphi$

$\Rightarrow$ 可解出 $q=\sqrt{\frac{2E}{m\omega^2}}\sin(\omega t+\varphi)$，$p=\sqrt{2mE}\cos(\omega t+\varphi)$

解2：　$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$.　　H不含时，设 $S=W-Et$.　H中 $p\to\frac{\partial S}{\partial q}=\frac{\partial W}{\partial q}$

$\Rightarrow H(q,W(q))=E$ 为 $\frac{1}{2m}(W'(q))^2+\frac{1}{2}m\omega^2q^2=E$.

$\to W'(q)=\sqrt{2m(E-\frac{1}{2}m\omega^2q^2}$.

$S(t,q,E)=\int dt\sqrt{2m(E-\frac{1}{2}m\omega^2q^2}-Et$.

不妨取新的动量为 $\alpha=E$　　　　量纲.

则 $\beta=\frac{\partial S}{\partial E}=\frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2E}}\omega q\right)-t=\text{const}.\ \overset{\downarrow}{\equiv}-t_0$

$\to$ 反解得 $q(t)=\sqrt{\frac{2E}{m\omega^2}}\sin(\omega(t-t_0))$

$p(t)=\frac{\partial S}{\partial q}=\sqrt{2mE-m^2\omega^2q^2}=\sqrt{2mE}\cos(\omega(t-t_0))$